THE BRUNN-MINKOWSKI INEQUALITY

OCTOBER 25, 2001

R. J. GARDNER

ABSTRACT. In 1978, Osserman [124] wrote a rather comprehensive survey on the isoperimetric inequality. The Brunn-Minkowski inequality can be proved in a page, yet quickly yields the classical isoperimetric inequality for important classes of subsets of $\mathbb{R}^n$, and deserves to be better known. We present a guide that explains the relationship between the Brunn-Minkowski inequality and other inequalities in geometry and analysis, and some of its recent applications.

1. INTRODUCTION

About a century ago, not long after the first complete proof of the classical isoperimetric inequality was found, Minkowski proved the following inequality:

$$V\left((1-\lambda)K+\lambda L\right)^{1/n} \geq (1-\lambda)V(K)^{1/n}+\lambda V(L)^{1/n}. \tag{1}$$

Here K and L are convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^n$, $0<\lambda<1$, V denotes volume, and $+$ denotes vector or Minkowski sum. The inequality (1) had been proved for $n=3$ earlier by Brunn, and now it is known as the *Brunn-Minkowski inequality.* It is a sharp inequality, equality holding if and only if K and L are homothetic.

The Brunn-Minkowski inequality was inspired by issues around the isoperimetric problem, and was for a long time considered to belong to geometry, where its significance is widely recognized. It implies, but is much stronger than, the intuitively clear fact that the function that gives the volumes of parallel hyperplane sections of a convex body is unimodal. It can be proved on a single page (see Section 6), yet it quickly yields the classical isoperimetric inequality (21) for convex bodies and other important classes of sets. The fundamental geometric content of the Brunn-Minkowski inequality makes it a cornerstone of the Brunn-Minkowski theory, a beautiful and powerful apparatus for conquering all sorts of problems involving metric quantities such as volume, surface area, and mean width.

By the mid-twentieth century, however, when Lusternik, Hadwiger and Ohmann, and Henstock and Macbeath had established a satisfactory generalization of (1) and its equality conditions to Lebesgue measurable sets, the inequality had begun its move into the realm of analysis. The last twenty years have seen the Brunn-Minkowski inequality consolidate its role as an analytical tool,

---

1991 *Mathematics Subject Classification.* 26D15, 52A40.

Key words and phrases. Brunn-Minkowski inequality, Minkowski's first inequality, Prékopa-Leindler inequality, Young's inequality, Brascamp-Lieb inequality, Barthe's inequality, isoperimetric inequality, Sobolev inequality, entropy power inequality, covariogram, Anderson's theorem, concave function, concave measure, convex body, mixed volume.

Supported in part by NSF Grant DMS 9802388.

and a compelling picture (see Figure 1) has emerged of its relations to other analytical inequalities. In an integral version of the Brunn-Minkowski inequality often called the Prékopa-Leindler inequality (12), a reverse form of Hölder's inequality, the geometry seems to have evaporated. Largely through the efforts of Brascamp and Lieb, this can be viewed as a special case of a sharp reverse form (32) of Young's inequality for convolution norms. A remarkable sharp inequality (36) proved by Barthe, closely related to (32), takes us up to the present time. The modern viewpoint entails an interaction between analysis and convex geometry so potent that whole conferences and books are devoted to "analytical convex geometry" or "convex geometric analysis."

The main development of this paper includes historical remarks and several detailed proofs that amplify the previous paragraph and show that even the latest developments are accessible to graduate students. Several applications are also discussed at some length. Extensions of the Prékopa-Leindler inequality can be used to obtain concavity properties of probability measures generated by densities of well-known distributions. Such results are related to Anderson's theorem on multivariate unimodality, an application of the Brunn-Minkowski inequality that in turn is useful in statistics. The entropy power inequality (48) of information theory has a form similar to that of the Brunn-Minkowski inequality. To some extent this is explained by Lieb's proof that the entropy power inequality is a special case of a sharp form of Young's inequality (31). This is given in detail along with some brief comments on the role of Fisher information and applications to physics. We come full circle with consequences of the later inequalities in convex geometry. Ball started these rolling with his elegant application of the Brascamp-Lieb inequality (35) to the volume of central sections of the cube and to a reverse isoperimetric inequality (45).

The whole story extends far beyond Figure 1 and the previous paragraph. The final Section 19 is a survey of the many other extensions, analogues, variants, and applications of the Brunn-Minkowski inequality. Essentially the strongest inequality for compact convex sets in the direction of the Brunn-Minkowski inequality is the Aleksandrov-Fenchel inequality (51). Here there is a remarkable link with algebraic geometry: Khovanskii and Teissier independently discovered that the Aleksandrov-Fenchel inequality can be deduced from the Hodge index theorem. Analogues and variants of the Brunn-Minkowski inequality include Borell's inequality (57) for capacity, employed in the recent solution of the Minkowski problem for capacity; Milman's reverse Brunn-Minkowski inequality (64), which features prominently in the local theory of Banach spaces; a discrete Brunn-Minkowski inequality (65) due to the author and Gronchi, closely related to a rich area of discrete mathematics, combinatorics, and graph theory concerning discrete isoperimetric inequalities; and inequalities (67), (68) originating in Busemann's theorem, motivated by his theory of area in Finsler spaces and used in Minkowski geometry and geometric tomography. Around the corner from the Brunn-Minkowski inequality lies a slew of related affine isoperimetric inequalities, such as the Petty projection inequality (62) and Zhang's affine Sobolev inequality (63), much more powerful than the isoperimetric inequality and the classical Sobolev inequality (24), respectively. There are versions of the Brunn-Minkowski inequality in the sphere, hyperbolic space, Minkowski spacetime, and Gauss space, and there is a Riemannian version of the Prékopa-Leindler inequality, obtained very recently by Cordero-Erausquin, McCann, and Schmuckenschläger. Finally, pointers are given to other applications of the Brunn-Minkowski inequality. Worthy of special mention here is the derivation of logarithmic Sobolev inequalities from the Prékopa-Leindler inequality by Bobkov and Ledoux, and work of Brascamp and Lieb, Borell, McCann, and others on diffusion equations. Measure-preserving convex gradients and transportation of mass, utlilized by McCann in applications to shapes of crystals and interacting gases, were also employed by Barthe in the proof of his inequality.

The reader might share a sense of mystery and excitement. In a sea of mathematics, the Brunn-Minkowski inequality appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next. It is quite clear that research opportunities abound. For example, what is the relationship between the Aleksandrov-Fenchel inequality and Barthe's inequality? Do even stronger inequalities await discovery in the region above Figure 1? Are there any hidden links between the various inequalities in Section 19? Perhaps, as more connections and relations are discovered, an underlying comprehensive theory will surface, one in which the classical Brunn-Minkowski theory represents just one particularly attractive piece of coral in a whole reef. Within geometry, the work of Lutwak and others in developing the dual Brunn-Minkowski and L^p-Brunn-Minkowski theories (see Section 19) strongly suggests that this might well be the case.

An early version of the paper was written to accompany a series of lectures given at the 1999 Workshop on Measure Theory and Real Analysis in Gorizia, Italy. I am very grateful to Franck Barthe, Apostolos Giannopoulos, Paolo Gronchi, Peter Gruber, Daniel Hug, Elliott Lieb, Robert McCann, Rolf Schneider, Béla Uhrin, and Gaoyong Zhang for their extensive comments on previous versions of this paper, as well as to many others who provided information and references.

2. A first step

An old saying has it that even a journey of a thousand miles must begin with a single step. Ours will be the following easy result (see Section 3 for definitions and notation).

Theorem 2.1. (Brunn-Minkowski inequality in $\mathbb{R}$.) *Let $0<\lambda<1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}$ such that $(1-\lambda)X+\lambda Y$ is also measurable. Then*

$$V_1\left((1-\lambda)X+\lambda Y\right) \geq (1-\lambda)V_1(X)+\lambda V_1(Y). \tag{2}$$

Proof. Suppose that X and Y are compact sets. It is straightforward to prove that $X+Y$ is also compact. Since the measures do not change, we can translate X and Y so that $X\cap Y=\{o\}$, $X\subset\{x : x\leq 0\}$, and $Y\subset\{x : x\geq 0\}$. Then $X+Y\supset X\cup Y$, so

$$V_1(X+Y)\geq V_1(X\cup Y)=V_1(X)+V_1(Y).$$

If we replace X by $(1-\lambda)X$ and Y by λY, we obtain (2) for compact X and Y. The general case follows easily by approximation from within by compact sets. □

Simple though it is, Theorem 2.1 already raises two important matters.

Firstly, observe that it was enough to prove the theorem when the factors $(1-\lambda)$ and λ are omitted. This is due to the positive homogeneity (of degree 1) of Lebesgue measure in $\mathbb{R}$: $V_1(rX)=rV_1(X)$ for $r\geq 0$. In fact, this property allows these factors to be replaced by arbitrary nonnegative real numbers. For reasons that will become clear, it will be convenient for most of the paper to incorporate the factors $(1-\lambda)$ and λ.

Secondly, the set $(1-\lambda)X+\lambda Y$ may not be measurable, even when X and Y are measurable. We discuss this point in more detail in Section 9.

The assumption in Theorem 2.1 and its n-dimensional forms, Theorem 5.1 and Corollary 5.3 below, that the sets are bounded is easily removed and is retained simply for convenience.

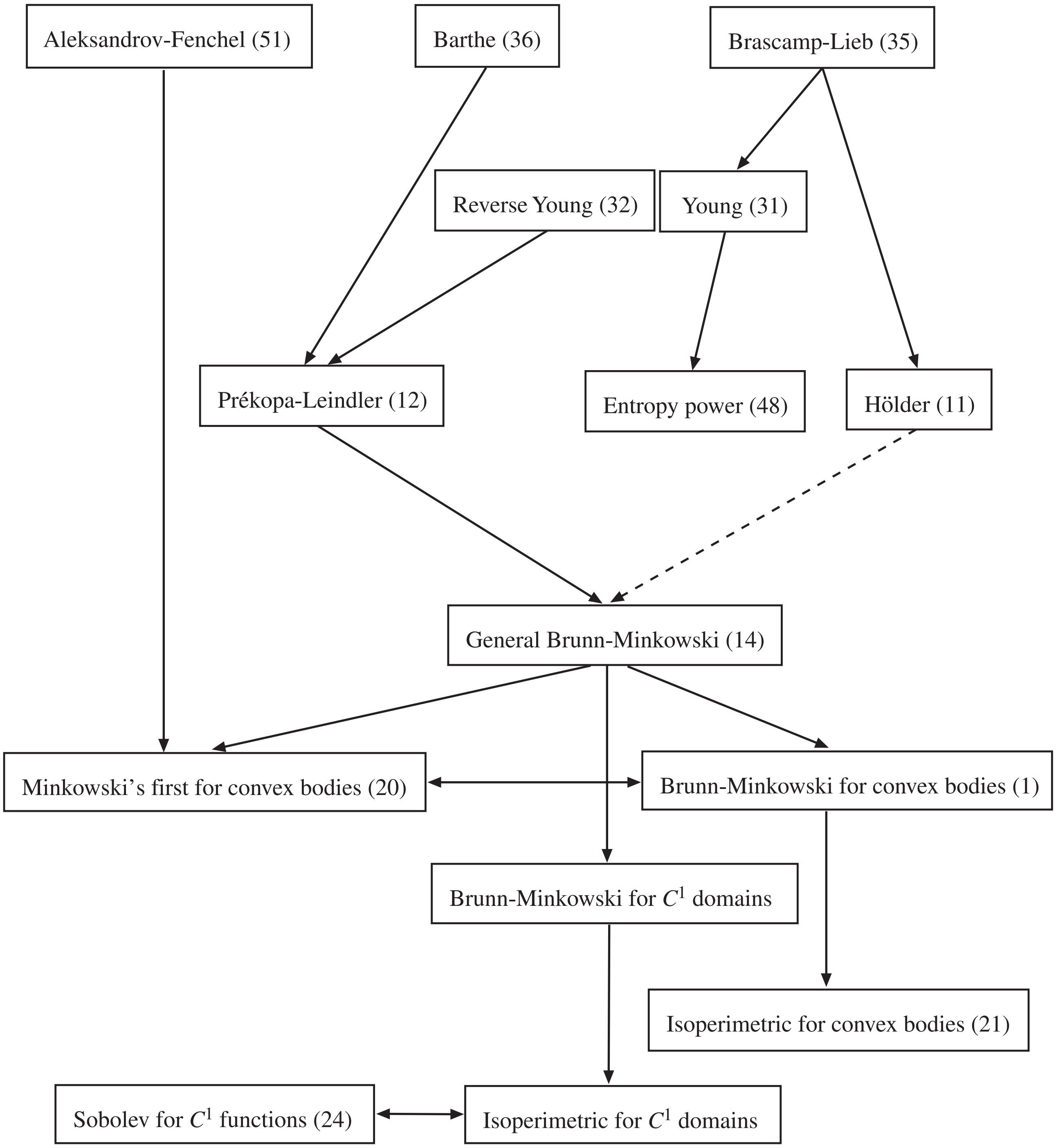

FIGURE 1. Relations between inequalities labeled as in the text.

3. A few preliminaries

We denote the origin, unit sphere, and closed unit ball in n-dimensional Euclidean space $\mathbb{R}^n$ by o, S^{n-1}, and B, respectively. The Euclidean scalar product of x and y will be written $x \cdot y$, and $\|x\|$ denotes the Euclidean norm of x. If $u \in S^{n-1}$, then $u^\perp$ is the hyperplane containing o and orthogonal to u.

Lebesgue k-dimensional measure V_k in $\mathbb{R}^n$, $k = 1, \dots, n$, can be identified with k-dimensional Hausdorff measure in $\mathbb{R}^n$. Then spherical Lebesgue measure in S^{n-1} can be identified with V_{n-1} in S^{n-1}. In this paper dx will denote integration with respect to V_k for the appropriate k and integration over S^{n-1} with respect to V_{n-1} will be denoted by du.

The term "measurable" applied to a set in $\mathbb{R}^n$ will mean V_n-measurable unless stated otherwise.

If X is a compact set in $\mathbb{R}^n$ with nonempty interior, we often write $V(X) = V_n(X)$ for its *volume*. We shall do this in particular when X is a *convex body*, a compact convex set with nonempty interior. We also write $\kappa_n = V(B)$. In geometry, it is customary to use the term *volume*, more generally, to mean the k-dimensional Lebesgue measure of a k-dimensional compact *body* X (equal to the closure of its relative interior), i.e. to write $V(X) = V_k(X)$ in this case.

Let X and Y be sets in $\mathbb{R}^n$. We define their *vector* or *Minkowski sum* by

$$X + Y = \{x + y : x \in X,\ y \in Y\}.$$

If $r \in \mathbb{R}$, let

$$rX = \{rx : x \in X\}.$$

If $r > 0$, then rX is the *dilatation* of X with factor r, and if $r < 0$, it is the reflection of this dilatation in the origin. If $0 < \lambda < 1$, the set $(1-\lambda)X + \lambda Y$ is called a *convex combination* of X and Y.

Minkowski's definition of the *surface area* $S(M)$ of a suitable set M in $\mathbb{R}^n$ is

$$S(M) = \lim_{\varepsilon \to 0+} \frac{V_n(M + \varepsilon B) - V_n(M)}{\varepsilon}. \tag{3}$$

In this paper we will use this definition when M is a convex body or a compact domain with piecewise C^1 boundary.

A function f on $\mathbb{R}^n$ is *concave* on a convex set C if

$$f\left((1-\lambda)x + \lambda y\right) \ge (1-\lambda)f(x) + \lambda f(y),$$

for all $x, y \in C$ and $0 < \lambda < 1$, and a function f is *convex* if $-f$ is concave. A nonnegative function f is *log concave* if $\log f$ is concave. Since the latter condition is equivalent to

$$f\left((1-\lambda)x + \lambda y\right) \ge f(x)^{1-\lambda} f(y)^\lambda,$$

the arithmetic-geometric mean inequality implies that each concave function is log concave.

If f is a nonnegative measurable function on $\mathbb{R}^n$ and $t \ge 0$, the *level set* $L(f,t)$ is defined by

$$L(f,t) = \{x : f(x) \ge t\}. \tag{4}$$

By Fubini's theorem,

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n}\int_0^{f(x)} 1\,dt\,dx = \int_0^\infty \int_{L(f,t)} 1\,dx\,dt = \int_0^\infty V_n\left(L(f,t)\right)\,dt. \tag{5}$$

If E is a set, 1_E denotes the *characteristic function* of E. The formula

$$f(x) = \int_0^\infty 1_{L(f,t)}(x)\,dt \tag{6}$$

follows easily from $f(x) = \int_0^{f(x)} dt$. In [91, Theorem 1.13], equation (6) is called the *layer cake representation* of f.

4. The Prékopa-Leindler inequality

Theorem 4.1. (Prékopa-Leindler inequality in $\mathbb{R}$.) *Let $0 < \lambda < 1$ and let f, g, and h be nonnegative integrable functions on $\mathbb{R}$ satisfying*

$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^\lambda, \tag{7}$$

for all $x, y \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} h(x)\,dx \geq \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}.$$

Two proofs of this fundamental result will be presented after a comment about the strange-looking assumption (7) that ensures h is not too small. Fix a $z \in \mathbb{R}$ and choose $0 < \lambda < 1$ and *any* $x, y \in \mathbb{R}$ such that $z = (1-\lambda)x + \lambda y$. Then the value of h at z must be at least the weighted geometric mean (it is the geometric mean if $\lambda = 1/2$) of the values of f at x and g at y. Note also that the logarithm of (7) yields the equivalent condition

$$\log h\left((1-\lambda)x + \lambda y\right) \geq (1-\lambda)\log f(x) + \lambda \log g(y).$$

If $f = g = h$, we would have

$$\log f\left((1-\lambda)x + \lambda y\right) \geq (1-\lambda)\log f(x) + \lambda \log f(y),$$

which just says that f is log concave. Of course, the previous theorem does not say anything when $f = g = h$.

First proof. We can assume without loss of generality that f and g are bounded with

$$\sup_{x\in\mathbb{R}} f(x) = \sup_{x\in\mathbb{R}} g(x) = 1.$$

If $t \geq 0$, $f(x) \geq t$, and $g(y) \geq t$, then by (7), $h\left((1-\lambda)x + \lambda y\right) \geq t$. With the notation (4) for level sets,

$$L(h,t) \supset (1-\lambda)L(f,t) + \lambda L(g,t),$$

for $0 \leq t < 1$. The sets on the right-hand side are nonempty, so by (5), the Brunn-Minkowski inequality (2) in $\mathbb{R}$, and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} h(x)\,dx &\geq \int_0^1 V_1\left(L(h,t)\right)\,dt \\
&\geq \int_0^1 V_1\left((1-\lambda)L(f,t)+\lambda L(g,t)\right)\,dt \\
&\geq (1-\lambda)\int_0^1 V_1\left(L(f,t)\right)\,dt + \lambda\int_0^\infty V_1\left(L(g,t)\right)\,dt \\
&= (1-\lambda)\int_{\mathbb{R}} f(x)\,dx + \lambda\int_{\mathbb{R}} g(x)\,dx \\
&\geq \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}.
\end{aligned}$$

□

Second proof. We can assume without loss of generality that

$$\int_{\mathbb{R}} f(x)\,dx = F > 0 \text{ and } \int_{\mathbb{R}} g(x)\,dx = G > 0.$$

Define $u, v : [0,1] \to \mathbb{R}$ such that $u(t)$ and $v(t)$ are the smallest numbers satisfying

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)\,dx = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)\,dx = t. \tag{8}$$

Then u and v may be discontinuous, but they are strictly increasing functions and so are differentiable almost everywhere. Let

$$w(t) = (1-\lambda)u(t) + \lambda v(t).$$

Take the derivative of (8) with respect to t to obtain

$$\frac{f\left(u(t)\right)u'(t)}{F} = \frac{g\left(v(t)\right)v'(t)}{G} = 1.$$

Using this and the arithmetic-geometric mean inequality, we obtain (when $f\left(u(t)\right) \neq 0$ and $g\left(u(t)\right) \neq 0$)

$$\begin{aligned}
w'(t) &= (1-\lambda)u'(t) + \lambda v'(t) \\
&\geq u'(t)^{1-\lambda}v'(t)^{\lambda} \\
&= \left(\frac{F}{f\left(u(t)\right)}\right)^{1-\lambda}\left(\frac{G}{g\left(v(t)\right)}\right)^{\lambda}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}} h(x)\,dx &\geq \int_0^1 h\left(w(t)\right)w'(t)\,dt \\
&\geq \int_0^1 f\left(u(t)\right)^{1-\lambda} g\left(v(t)\right)^{\lambda}\left(\frac{F}{f\left(u(t)\right)}\right)^{1-\lambda}\left(\frac{G}{g\left(v(t)\right)}\right)^{\lambda}\,dt = F^{1-\lambda}G^{\lambda}.
\end{aligned}$$

□

There are two basic ingredients in the second proof of Theorem 4.1: the introduction in (8) of the volume parameter t, and use of the arithmetic-geometric mean inequality in estimating $w'(t)$. The same ingredients appear in the first proof, though the parametrization is somewhat disguised in the use of the level sets.

Theorem 4.2. (Prékopa-Leindler inequality in $\mathbb{R}^n$.) *Let $0 < \lambda < 1$ and let f, g, and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying*

$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^{\lambda}, \tag{9}$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,dx \geq \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx\right)^{\lambda}.$$

Proof. The proof is by induction on n. It is true for $n = 1$, by Theorem 4.1. Suppose that it is true for all natural numbers less than n.

For each $s \in \mathbb{R}$, define a nonnegative function h_s on $\mathbb{R}^{n-1}$ by $h_s(z) = h(z, s)$ for $z \in \mathbb{R}^{n-1}$, and define f_s and g_s analogously. Let $x, y \in \mathbb{R}^{n-1}$, let $a, b \in \mathbb{R}$, and let $c = (1-\lambda)a + \lambda b$. Then

$$\begin{aligned} h_c\left((1-\lambda)x + \lambda y\right) &= h\left((1-\lambda)x + \lambda y, (1-\lambda)a + \lambda b\right) \\ &= h\left((1-\lambda)(x, a) + \lambda(y, b)\right) \\ &\geq f(x, a)^{1-\lambda} g(y, b)^{\lambda} \\ &= f_a(x)^{1-\lambda} g_b(y)^{\lambda}. \end{aligned}$$

By the inductive hypothesis,

$$\int_{\mathbb{R}^{n-1}} h_c(x)\,dx \geq \left(\int_{\mathbb{R}^{n-1}} f_a(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_b(x)\,dx\right)^{\lambda}.$$

Let

$$H(c) = \int_{\mathbb{R}^{n-1}} h_c(x)\,dx, \ F(a) = \int_{\mathbb{R}^{n-1}} f_a(x)\,dx, \text{ and } G(b) = \int_{\mathbb{R}^{n-1}} g_b(x)\,dx.$$

Then

$$H(c) = H\left((1-\lambda)a + \lambda b\right) \geq F(a)^{1-\lambda} G(b)^{\lambda}.$$

So, by Fubini's theorem and Theorem 4.1,

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)\,dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} h_c(z)\,dz\,dc \\ &= \int_{\mathbb{R}} H(c)\,dc \\ &\geq \left(\int_{\mathbb{R}} F(a)\,da\right)^{1-\lambda} \left(\int_{\mathbb{R}} G(b)\,db\right)^{\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx\right)^{\lambda}. \end{aligned}$$

□

Suppose that $f_i \in L^{p_i}(\mathbb{R}^n)$, $p_i \geq 1$, $i = 1, \ldots, m$ are nonnegative functions, where

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1. \tag{10}$$

Hölder's inequality in $\mathbb{R}^n$ states that

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x)\,dx \leq \prod_{i=1}^m \|f_i\|_{p_i} = \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(x)^{p_i}\,dx\right)^{1/p_i}. \tag{11}$$

Let $0 < \lambda < 1$. If $m = 2$, $1/p_1 = 1 - \lambda$, $1/p_2 = \lambda$, and we let $f = f_1^{p_1}$ and $g = f_2^{p_2}$, we get

$$\int_{\mathbb{R}^n} f(x)^{1-\lambda} g(x)^\lambda\,dx \leq \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx\right)^\lambda.$$

The Prékopa-Leindler inequality in $\mathbb{R}^n$ can be written in the form

$$\overline{\int}_{\mathbb{R}^n} \sup\{f(x)^{1-\lambda} g(y)^\lambda : (1-\lambda)x + \lambda y = z\}\,dz \geq \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx\right)^\lambda, \tag{12}$$

because we can use the supremum for h in (9). A straightforward generalization is

$$\overline{\int}_{\mathbb{R}^n} \sup\left\{\prod_{i=1}^m f_i(x_i) : \sum_{i=1}^m \frac{x_i}{p_i} = z\right\} dz \geq \prod_{i=1}^m \|f_i\|_{p_i}, \tag{13}$$

where $p_i \geq 1$ for each i and (10) holds. So we see that the Prékopa-Leindler inequality is a reverse form of Hölder's inequality and that some condition such as (7) is therefore necessary for it to hold.

Notice that the upper Lebesgue integral is used on the left in (12) and (13). This is because the integrands there are generally not measurable. We shall return to this point in Section 9.

5. The Brunn-Minkowski inequality

In this section the Brunn-Minkowski inequality is derived from the Prékopa-Leindler inequality. A different and self-contained short proof can be found in Section 6.

Theorem 5.1. (General Brunn-Minkowski inequality in $\mathbb{R}^n$, first form.) *Let $0 < \lambda < 1$ and let X and Y be bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)X + \lambda Y$ is also measurable. Then*

$$V_n((1-\lambda)X + \lambda Y) \geq V_n(X)^{1-\lambda} V_n(Y)^\lambda. \tag{14}$$

Theorem 5.2. *The Prékopa-Leindler inequality in $\mathbb{R}^n$ implies the general Brunn-Minkowski inequality in $\mathbb{R}^n$.*

Proof. Let $h = 1_{(1-\lambda)X+\lambda Y}$, $f = 1_X$, and $g = 1_Y$. If $x, y \in \mathbb{R}^n$, then $f(x)^{1-\lambda} g(y)^\lambda > 0$ (and in fact equals 1) if and only if $x \in X$ and $y \in Y$. The latter implies $(1-\lambda)x + \lambda y \in (1-\lambda)X + \lambda Y$, which is true if and only if $h((1-\lambda)x + \lambda y) = 1$. Therefore (9) holds. We conclude by Theorem 4.2 that

$$\begin{aligned} V_n((1-\lambda)X + \lambda Y) &= \int_{\mathbb{R}^n} 1_{(1-\lambda)X+\lambda Y}(x)\,dx \\ &\geq \left(\int_{\mathbb{R}^n} 1_X(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} 1_Y(x)\,dx\right)^\lambda = V_n(X)^{1-\lambda} V_n(Y)^\lambda. \end{aligned}$$

□

Corollary 5.3. (General Brunn-Minkowski inequality in $\mathbb{R}^n$, standard form.) *Let* $0 < \lambda < 1$ *and let* X *and* Y *be nonempty bounded measurable sets in* $\mathbb{R}^n$ *such that* $(1-\lambda)X + \lambda Y$ *is also measurable. Then*

$$V_n\left((1-\lambda)X + \lambda Y\right)^{1/n} \geq (1-\lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}. \tag{15}$$

Proof. Let

$$\lambda' = \frac{V_n(Y)^{1/n}}{V_n(X)^{1/n} + V_n(Y)^{1/n}}$$

and let $X' = V_n(X)^{-1/n}X$ and $Y' = V_n(Y)^{-1/n}Y$. Then

$$1 - \lambda' = \frac{V_n(X)^{1/n}}{V_n(X)^{1/n} + V_n(Y)^{1/n}}$$

and $V_n(X') = V_n(Y') = 1$, by the positive homogeneity (of degree n) of Lebesgue measure in $\mathbb{R}^n$ ($V_n(rA) = r^n V_n(A)$ for $r \geq 0$). Therefore (14), applied to X', Y', and λ', yields

$$V_n\left((1-\lambda')X' + \lambda' Y'\right) \geq 1.$$

But

$$V_n\left((1-\lambda')X' + \lambda' Y'\right) = V_n\left(\frac{X+Y}{V_n(X)^{1/n} + V_n(Y)^{1/n}}\right) = \frac{V_n(X+Y)}{\left(V_n(X)^{1/n} + V_n(Y)^{1/n}\right)^n}.$$

This gives

$$V_n(X+Y)^{1/n} \geq V_n(X)^{1/n} + V_n(Y)^{1/n}.$$

To obtain (15), just replace X and Y by $(1-\lambda)X$ and λY, respectively. □

Remark 5.4. Using the homogeneity of volume, it follows that for all $s, t > 0$,

$$V_n\left(sX + tY\right)^{1/n} \geq sV_n(X)^{1/n} + tV_n(Y)^{1/n}. \tag{16}$$

Note the advantages of the first form (14) of the general Brunn-Minkowski inequality. One need not assume that X and Y are nonempty, and the inequality is independent of the dimension n. The two forms are equivalent, however; to get from the standard to the first form, just use Jensen's inequality for means (see (28) below with $p = 0$ and $q = 1/n$).

6. History, alternative proofs, and equality conditions

For detailed remarks and references concerning the early history of the Brunn-Minkowski inequality for convex bodies, see [134, p. 314]. Briefly, the inequality for convex bodies in $\mathbb{R}^3$ was discovered by Brunn around 1887. Minkowski pointed out an error in the proof, which Brunn corrected, and found a different proof himself. Both Brunn and Minkowski showed that equality holds if and only if K and L are *homothetic* (i.e., K and L are equal up to translation and dilatation). The proof presented in [134, Section 6.1], due to Kneser and Süss in 1932, is very similar to the proof we gave above of the Prékopa-Leindler inequality, restricted to characteristic functions of convex bodies; note that the case $n = 1$ is trivial, and the equality condition vacuous, in this case. This is perhaps the simplest approach for the equality conditions for convex bodies.

Another quite different proof, due to Blaschke in 1917, is worth mentioning. This uses Steiner symmetrization. Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. The *Steiner symmetral* S_uK of K in the direction u is the convex body obtained from K by sliding each of its chords parallel to u so that they are bisected by the hyperplane $u^\perp$, and taking the union of the resulting chords. Then $V(S_uK) = V(K)$ by Cavalieri's principle, and it is not hard to show that if K and L are convex bodies in $\mathbb{R}^n$, then

$$S_u(K+L) \supset S_uK + S_uL. \tag{17}$$

One can also prove that there is a sequence of directions $u_m \in S^{n-1}$ such that if K is any convex body and $K_m = S_{u_m}K_{m-1}$, then $K_m \to r_KB$ as $m \to \infty$, where r_K is the constant such that $V(K) = V(r_KB)$. Repeated application of (17) now gives

$$\begin{aligned} V(K+L)^{1/n} &\geq V(r_KB + r_LB)^{1/n} = (r_R + r_L)V(B)^{1/n} \\ &= V(r_KB)^{1/n} + V(r_LB)^{1/n} = V(K)^{1/n} + V(L)^{1/n}. \end{aligned}$$

See [53, Chapter 5, Section 5] or [150, pp. 310–314] for more details.

The general Brunn-Minkowski inequality and its equality conditions were first proved by Lusternik [94]. The equality conditions he gave were corrected by Henstock and Macbeath [78], who basically used the method in the second proof of Theorem 4.1 to derive the inequality. Another method, found by Hadwiger and Ohmann [76], is so beautiful that we cannot resist reproducing it in full (see also [37, Section 8], [51, Section 6.6], [58, Theorem 3.2.41], or [150, Section 6.5]). The idea is to prove the result first for boxes, rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes. If X and Y are boxes with sides of length x_i and y_i, respectively, in the ith coordinate directions, then

$$V(X) = \prod_{i=1}^n x_i, \;\; V(Y) = \prod_{i=1}^n y_i, \;\text{ and } V(X+Y) = \prod_{i=1}^n (x_i + y_i).$$

Now

$$\left(\prod_{i=1}^n \frac{x_i}{x_i+y_i}\right)^{1/n} + \left(\prod_{i=1}^n \frac{y_i}{x_i+y_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n \frac{x_i}{x_i+y_i} + \frac{1}{n}\sum_{i=1}^n \frac{y_i}{x_i+y_i} = 1,$$

by the arithmetic-geometric mean inequality. This gives the Brunn-Minkowski inequality for boxes. One then uses a trick sometimes called a *Hadwiger-Ohmann cut* to obtain the inequality for finite unions X and Y of boxes, as follows. By translating X, if necessary, we can assume that a coordinate hyperplane, $\{x_n = 0\}$ say, separates two boxes in X. Let X_+ (or X_-) denote the union of the boxes formed by intersecting the boxes in X with $\{x_n \geq 0\}$ (or $\{x_n \leq 0\}$, respectively). Now translate Y so that

$$\frac{V(X_\pm)}{V(X)} = \frac{V(Y_\pm)}{V(Y)}, \tag{18}$$

where Y_+ and Y_- are defined analogously to X_+ and X_-. Note that $X_+ + Y_+ \subset \{x_n \geq 0\}$, $X_- + Y_- \subset \{x_n \leq 0\}$, and that the numbers of boxes in $X_+ \cup Y_+$ and $X_- \cup Y_-$ are both smaller than the number of boxes in $X \cup Y$. By induction on the latter number and (18), we have

$$\begin{aligned} V(X+Y) &\geq V(X_+ + Y_+) + V(X_- + Y_-) \\ &\geq \left(V(X_+)^{1/n} + V(Y_+)^{1/n}\right)^n + \left(V(X_-)^{1/n} + V(Y_-)^{1/n}\right)^n \\ &= V(X_+)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n + V(X_-)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n \\ &= V(X)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n = \left(V(X)^{1/n} + V(Y)^{1/n}\right)^n. \end{aligned}$$

Now that the inequality is established for finite unions of boxes, the proof is completed by using them to approximate bounded measurable sets. A careful examination of this proof allows one to conclude that if $V_n(X)V_n(Y) > 0$, equality holds only when

$$V_n\left((\operatorname{conv} X) \setminus X\right) = V_n\left((\operatorname{conv} Y) \setminus Y\right) = 0,$$

where $\operatorname{conv} X$ denotes the *convex hull* of X. Putting the equality conditions above together, we see that *if $V_n(X)V_n(Y) > 0$, equality holds in the general Brunn-Minkowski inequality (5.1) or (5.3) if and only if X and Y are homothetic convex bodies from which sets of measure zero have been removed*. See [37, Section 8] and [150, Section 6.5] for more details and further comments about the case when X or Y has measure zero.

Since Hölder's inequality (11) in its discrete form implies the arithmetic-geometric mean inequality, there is a sense in which Hölder's inequality implies the Brunn-Minkowski inequality. The dotted arrow in Figure 1 reflects the controversial nature of this implication.

Though the Hadwiger-Ohmann proof looks quite different from the Henstock-Macbeath approach, it shares the same two basic ingredients mentioned after Theorem 4.1, since the Hadwiger-Ohmann cut (18) is tantamount to a parametrization by volume.

The Prékopa-Leindler inequality was explicitly stated and proved by Prékopa [128], [129] and Leindler [87]. (See also the historical remarks after Theorem 10.2, however.) The first proof of Theorem 4.1 presented here, which follows that of Brascamp and Lieb [35, Theorem 3.1], is also reproduced in [127, Theorem 1.1]. The parametrization idea in the second proof of Theorem 4.1 goes back to Bonnesen; see [47] and the references given there. The induction in Theorem 4.2 can be avoided and the inequality proved at once in $\mathbb{R}^n$ by means of the so-called *Knothe map* (see [120, p. 186]).

Quite complicated equality conditions for the Prékopa-Leindler inequality in $\mathbb{R}$ are given in [45] and [146], but equality conditions in $\mathbb{R}^n$ seem to be unknown.

Recently, Borell found a "Brownian motion" proof of the Brunn-Minkowski inequality that depends on a generalization of the Prékopa-Leindler inequality, too complicated to be stated here, involving diffusion equations; see [32] and also Section 19.14.

7. Minkowski's first inequality and the isoperimetric inequality

For some classes of sets such as convex bodies, the Brunn-Minkowski inequality is equivalent to another inequality of Minkowski that immediately yields the isoperimetric inequality. This involves a quantity $V_1(K, L)$ depending on two convex bodies K and L in $\mathbb{R}^n$ that can be defined by

$$nV_1(K,L) = \lim_{\varepsilon\to 0+} \frac{V(K+\varepsilon L)-V(K)}{\varepsilon}. \tag{19}$$

Note that if $L = B$, then $S(K) = nV_1(K,B)$; it is this relationship that will quickly lead us to the isoperimetric inequality and its equality condition. An even shorter path (see [67, Theorem B.2.1]) yields the inequality but without the equality condition.

The quantity $V_1(K,L)$ is a special mixed volume, and its existence requires just a little of the theory of mixed volumes to establish; see [150, Section 6.4]. In fact, Minkowski showed that if $K_1,\dots,K_m$ are compact convex sets in $\mathbb{R}^n$, and $t_1,\dots,t_m \ge 0$, the volume $V(\sum\{t_iK_i : i = 1,\dots,m\})$ is a polynomial of degree n in the variables $t_1,\dots,t_m$. The coefficient $V(K_{j_1},\dots,K_{j_n})$ of $t_{j_1}\cdots t_{j_n}$ in this polynomial is called a *mixed volume*. Then $V_1(K,L) = V(K,n-1;L)$, where the notation means that K appears $(n-1)$ times and L appears once. See [67, Appendix A] for a gentle introduction to mixed volumes.

Theorem 7.1. (Minkowski's first inequality for convex bodies in $\mathbb{R}^n$.) *Let K and L be convex bodies in $\mathbb{R}^n$. Then*

$$V_1(K,L) \ge V(K)^{(n-1)/n}V(L)^{1/n}, \tag{20}$$

with equality if and only if K and L are homothetic.

Minkowski's first inequality plays a role in the solution of Shephard's problem: If the projection of a centrally symmetric (i.e., $-K$ is a translate of K) convex body onto any given hyperplane is always smaller in volume than that of another such body, is its volume also smaller? The answer is no in general in three or more dimensions; see [67, Chapter 4] and [99, p. 255].

Theorem 7.2. *The Brunn-Minkowski inequality for convex bodies in $\mathbb{R}^n$ (and its equality condition) implies Minkowski's first inequality for convex bodies in $\mathbb{R}^n$ (and its equality condition).*

Proof. Substituting $\varepsilon = t/(1-t)$ in (19) and using the homogeneity of volume, we obtain

$$\begin{aligned} nV_1(K,L) &= \lim_{\varepsilon\to 0+} \frac{V(K+\varepsilon L)-V(K)}{\varepsilon} \\ &= \lim_{t\to 0+} \frac{V\left((1-t)K+tL\right)-(1-t)^nV(K)}{t(1-t)^{n-1}} \\ &= \lim_{t\to 0+} \frac{V\left((1-t)K+tL\right)-V(K)}{t} + \lim_{t\to 0+}\frac{(1-(1-t)^n))V(K)}{t} \\ &= \lim_{t\to 0+} \frac{V\left((1-t)K+tL\right)-V(K)}{t} + nV(K). \end{aligned}$$

Using this new expression for $V_1(K,L)$ (see [107, p. 7]) and letting $f(t) = V\left((1-t)K+tL\right)^{1/n}$, for $0 \le t \le 1$, we see that

$$f'(0) = \frac{V_1(K,L)-V(K)}{V(K)^{(n-1)/n}}.$$

Therefore (20) is equivalent to $f'(0) \ge f(1) - f(0)$. Since the Brunn-Minkowski inequality says that f is concave, Minkowski's first inequality follows.

Suppose that equality holds in (20). Then $f'(0) = f(1) - f(0)$. Since f is concave, we have

$$\frac{f(t) - f(0)}{t} = f(1) - f(0)$$

for $0 < t \leq 1$, and this is just equality in the Brunn-Minkowski inequality. The equality condition for (20) follows immediately. $\square$

The following corollary is obtained by taking $L = B$ in Theorem 7.1.

Corollary 7.3. (Isoperimetric inequality for convex bodies in $\mathbb{R}^n$.) *Let K be a convex body in $\mathbb{R}^n$. Then*

$$\left(\frac{V(K)}{V(B)}\right)^{1/n} \leq \left(\frac{S(K)}{S(B)}\right)^{1/(n-1)}, \tag{21}$$

with equality if and only if K is a ball.

It can be shown (see [152]) that if M is a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary and L is a convex body in $\mathbb{R}^n$, the quantity $V_1(M, L)$ defined by (19) with K replaced by M exists. From the Brunn-Minkowski inequality for compact domains in $\mathbb{R}^n$ with piecewise C^1 boundary and the above argument, one obtains Minkowski's first inequality when the convex body K is replaced by such a domain. Taking $L = B$, this immediately gives the isoperimetric inequality for compact domains in $\mathbb{R}^n$ with piecewise C^1 boundary.

Essentially the most general class of sets for which the isoperimetric inequality in $\mathbb{R}^n$ is known to hold comprises the sets of finite perimeter; see, for example, the book of Evans and Gariepy [57, p. 190], where the rather technical setting, sometimes called the BV theory, is expounded. It is still possible to base the proof on the Brunn-Minkowski inequality, as Fonseca [62, Theorem 4.2] demonstrates, by first obtaining the isoperimetric inequality for suitably smooth sets and then applying various measure-theoretic approximation arguments. In fact, Fonseca's result is more general (see the material in Section 19.14 on Wulff shape of crystals). A strong form of the Brunn-Minkowski inequality is also used by Fonseca and Müller [63], again in the more general context of Wulff shape, to establish the corresponding equality conditions (the same as for (21)).

There is (see [134, Theorem 5.1.6] and [152]) an integral representation for mixed volumes, and in particular,

$$V_1(M, L) = \frac{1}{n} \int_{\partial M} h_L(u_x)\, dx, \tag{22}$$

where h_L is the support function of L and u_x is the outer unit normal vector to ∂M at x. (If we replace h_L by an arbitrary function f on S^{n-1}, then up to a constant, this integral represents the surface energy of a crystal with shape M, where f is the surface tension; see Section 19.14.) When $M = K$ is a sufficiently smooth convex body, (22) can be written

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) f_K(u)\, du, \tag{23}$$

where f_K is the reciprocal of the Gauss curvature of K at the point on ∂K where the outer unit normal is u; for general convex bodies, $f_K(u)\, du$ must be replaced by $dS(K, u)$, where $S(K, \cdot)$ is the *surface area measure* of K. Minkowski's existence theorem gives necessary and sufficient conditions for a measure μ in S^{n-1} to be the surface area measure of some convex body. Now (20) and (23) imply that if $S(K,\cdot)=\mu$, then K minimizes the functional

$$L \to \int_{S^{n-1}} h_L(u)\,d\mu$$

under the condition that $V(L)=1$, and this fact motivates the proof of Minkowski's existence theorem. See [134, Section 7.1], where pointers can also be found to the vast literature surrounding the so-called *Minkowski problem*, which deals with existence, uniqueness, regularity, and stability of a closed convex hypersurface whose Gauss curvature is prescribed as a function of its outer normals.

8. The Sobolev inequality

Theorem 8.1. (Sobolev inequality.) *Let f be a C^1 function on $\mathbb{R}^n$ with compact support. Then*

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|\,dx \ge n\kappa_n^{1/n}\|f\|_{n/(n-1)}. \tag{24}$$

The previous inequality is only one of a family, all called Sobolev inequalities. See [91, Chapter 8], where it is pointed out that such inequalities bound averages of gradients from below by weighted averages of the function, and can thus be considered as uncertainty principles.

Theorem 8.2. *The Sobolev inequality is equivalent to the isoperimetric inequality for compact domains with C^1 boundaries.*

Proof. Suppose that the isoperimetric inequality holds, and let f be a C^1 function on $\mathbb{R}^n$ with compact support. The coarea formula (a sort of curvilinear Fubini theorem; see [57, p. 112]) implies that

$$\begin{aligned}\int_{\mathbb{R}^n} \|\nabla f(x)\|\,dx &= \int_{\mathbb{R}} V_{n-1}(f^{-1}\{t\})\,dt\\ &= \int_0^\infty S\left(L(|f|,t)\right)\,dt,\end{aligned}$$

where $L(|f|,t)$ is a level set of $|f|$, as in (4). Applying the the isoperimetric inequality for compact domains with C^1 boundaries to these level sets, we obtain

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|\,dx \ge n\kappa_n^{1/n}\int_0^\infty V\left(L(|f|,t)\right)^{(n-1)/n}\,dt.$$

On the other hand, by (6) and Minkowski's inequality for integrals (see [77, (6.13.9), p. 148]), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f(x)|^{n/(n-1)}\,dx\right)^{(n-1)/n} &= \left(\int_{\mathbb{R}^n}\left(\int_0^\infty 1_{L(|f|,t)}(x)\,dt\right)^{n/(n-1)}dx\right)^{(n-1)/n}\\ &\le \int_0^\infty \left(\int_{\mathbb{R}^n} 1_{L(|f|,t)}(x)^{n/(n-1)}\,dx\right)^{(n-1)/n}dt\\ &= \int_0^\infty V\left(L(|f|,t)\right)^{(n-1)/n}\,dt.\end{aligned}$$

Therefore (24) is true.

Suppose that (24) holds, let M be a compact domain in $\mathbb{R}^n$ with C^1 boundary ∂M, and let $\varepsilon > 0$. Define $f_\varepsilon(x) = 1$ if $x \in M$, $f_\varepsilon(x) = 0$ if $x \notin M + \varepsilon B$, and $f_\varepsilon(x) = 1 - d(x, M)/\varepsilon$ if $x \in (M + \varepsilon B) \setminus M$, where $d(x, M)$ is the distance from x to M. Since f_ε can be approximated by C^1 functions on $\mathbb{R}^n$ with compact support, we can assume that (24) holds for f_ε. Note that $f_\varepsilon \to 1_M$ as $\varepsilon \to 0$. Also, $\|\nabla f_\varepsilon(x)\| = 1/\varepsilon$ if $x \in (M + \varepsilon B) \setminus M$ and is zero otherwise. Therefore, by (3),

$$\begin{aligned} S(M) = \lim_{\varepsilon \to 0+} \frac{V(M+\varepsilon B) - V(M)}{\varepsilon} &= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} \|\nabla f_\varepsilon(x)\|\, dx \\ &\geq \lim_{\varepsilon \to 0+} n\kappa_n^{1/n} \left(\int_{\mathbb{R}^n} |f_\varepsilon(x)|^{n/(n-1)}\, dx \right)^{(n-1)/n} \\ &= n\kappa_n^{1/n} \left(\int_{\mathbb{R}^n} 1_M(x)\, dx \right)^{(n-1)/n} \\ &= n\kappa_n^{1/n} V(M)^{(n-1)/n}, \end{aligned}$$

which is just a reorganization of the isoperimetric inequality (21) . □

As for the isoperimetric inequality, there is a more general version of the Sobolev inequality in the BV theory. This is called the Gagliardo-Nirenberg-Sobolev inequality and it is equivalent to the isoperimetric inequality for sets of finite perimeter; see [57, pp. 138 and 192].

The Prékopa-Leindler inequality can also be used to obtain logarithmic Sobolev inequalities; see Section 19.14.

9. Measurability in Brunn-Minkowski and Prékopa-Leindler

If X and Y are Borel sets, then $(1-\lambda)X + \lambda Y$, being a continuous image of their product, is analytic and hence measurable. (Erdös and Stone [56] proved that this set need not itself be Borel.) However, an old example of Sierpiński [137] shows that the set $(1-\lambda)X + \lambda Y$ may not be measurable when X and Y are measurable.

There are a couple of ways around the measurability problem. One can simply replace the measure on the left of the Brunn-Minkowski inequality by inner Lebesgue measure V_{n*}, the supremum of the measures of compact subsets, thus:

$$V_{n*}\left((1-\lambda)X + \lambda Y\right)^{1/n} \geq (1-\lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}.$$

A better solution is to obtain a slightly improved version of the Prékopa-Leindler inequality, and then deduce a corresponding improved Brunn-Minkowski inequality, as follows.

Recall that the *essential supremum* of a measurable function f on $\mathbb{R}^n$ is defined by

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^n} f(x) = \inf\{t : f(x) \leq t \text{ for almost all } x \in \mathbb{R}^n\}.$$

Brascamp and Lieb [35] proved the following result. (According to Uhrin [146], the idea of using the essential supremum in connection with our topic occurred independently to S. Dancs.)

Theorem 9.1. (Prékopa-Leindler inequality in $\mathbb{R}^n$, essential form.) *Let $0 < \lambda < 1$ and let $f, g \in L^1(\mathbb{R}^n)$ be nonnegative. Let*

$$s(x) = \operatorname{ess\,sup}_y f\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^{\lambda}. \tag{25}$$

Then s is measurable and

$$\|s\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

Proof. First note that s is measurable. Indeed,

$$s(x) = \sup_{\phi \in D} \int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^{\lambda} \phi(y)\,dy,$$

where D is a countable dense subset of the unit ball of $L^1(\mathbb{R}^n)$. Therefore s is the supremum of a countable family of measurable functions.

With the measurability of s in hand, the proof follows that of the usual Prékopa-Leindler inequality presented in Section 4. □

The essential form of the Prékopa-Leindler inequality in $\mathbb{R}^n$ implies the usual form, Theorem 4.2. To see this, replace x by z and y by $\lambda y'$ in (25) and then let $x = (z - \lambda y')/(1-\lambda)$ to obtain

$$\begin{aligned} s(z) &= \operatorname{ess\,sup}_{y'} f\left(\frac{z-\lambda y'}{1-\lambda}\right)^{1-\lambda} g(y')^{\lambda} \\ &= \operatorname{ess\,sup}\{f(x)^{1-\lambda} g(y)^{\lambda} : \ z = (1-\lambda)x + \lambda y\}. \end{aligned}$$

Now if h is any integrable function satisfying

$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^{\lambda},$$

we must have $h \geq s$ almost everywhere. It follows from Theorem 9.1 that

$$\|h\|_1 \geq \|s\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

The corresponding improvement of the Brunn-Minkowski inequality requires one new concept. Note that the usual Minkowski sum of X and Y can be written

$$X + Y = \{z : X \cap (z - Y)\} \neq \emptyset.$$

Adjust this by defining the *essential sum* of X and Y by

$$X +_e Y = \{z : V_n\left(X \cap (z - Y)\right) > 0\}.$$

While

$$1_{X+Y}(z) = \sup_{x \in \mathbb{R}^n} 1_X(x) 1_Y(z - x),$$

it is easy to see that

$$1_{X +_e Y}(z) = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} 1_X(x) 1_Y(z - x). \tag{26}$$

Theorem 9.2. (General Brunn-Minkowski inequality in $\mathbb{R}^n$, essential form.) *Let $0 < \lambda < 1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}^n$. Then*

$$V_n\left((1-\lambda)X +_e \lambda Y\right)^{1/n} \geq (1-\lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}. \tag{27}$$

Proof. In Theorem 9.1, let $f = 1_{(1-\lambda)X}$ and $g = 1_{\lambda Y}$. Then, by (26),

$$\begin{aligned} 1_{(1-\lambda)X+_e\lambda Y}(z) &= \operatorname{ess}\sup_{x\in\mathbb{R}^n} 1_{(1-\lambda)X}(x)1_{\lambda Y}(z-x) \\ &= \operatorname{ess}\sup_{x\in\mathbb{R}^n} 1_X\left(\frac{x}{1-\lambda}\right)1_Y\left(\frac{z-x}{\lambda}\right) \\ &= \operatorname{ess}\sup_{y\in\mathbb{R}^n} 1_X\left(\frac{z-y}{1-\lambda}\right)1_Y\left(\frac{y}{\lambda}\right) = s(z). \end{aligned}$$

The inequality

$$V_n\left((1-\lambda)X +_e \lambda Y\right) \geq V_n(X)^{1-\lambda}V_n(Y)^{\lambda},$$

and hence (27), now follow exactly as in Section 5. □

A direct proof of the previous theorem is given in [35, Appendix]. Here is a sketch. One first shows that $X +_e Y$ is measurable (indeed, open). This is proved using the set A^* of density points of a measurable set A, that is,

$$A^* = \left\{x \in \mathbb{R}^n : \lim_{\varepsilon\to 0+} \frac{V_n\left(A\cap B(x,\varepsilon)\right)}{V_n(B(x,\varepsilon))} = 1\right\},$$

where $B(x,\varepsilon)$ is a ball with center at x and radius ε. Then $V_n(A\triangle A^*) = 0$, where $\triangle$ denotes symmetric difference, and this implies that

$$X +_e Y = X^* +_e Y^*.$$

Now it can be shown that $X^* +_e Y^*$ is open and

$$X^* +_e Y^* = X^* + Y^*.$$

The Brunn-Minkowski inequality (15) in $\mathbb{R}^n$ then implies (27).

10. p-concave functions and measures

If f is a nonnegative integrable function defined on a measurable subset A of $\mathbb{R}^n$, and μ is defined by

$$\mu(X) = \int_{A\cap X} f(x)\,dx,$$

for all measurable subsets X of $\mathbb{R}^n$, we say that μ is *generated* by f and A.

The Prékopa-Leindler inequality implies that if f is log concave and C is an open convex subset of its support, then the measure μ generated by f and C is also log concave. Indeed, if $0 < \lambda < 1$, X and Y are measurable sets, and $z = (1-\lambda)x + \lambda y$, then the log concavity of f implies

$$f(z)1_{C\cap((1-\lambda)X+\lambda Y)}(z) \geq \left(f(x)1_{C\cap X}(x)\right)^{1-\lambda}\left(f(y)1_{C\cap Y}(y)\right)^{\lambda},$$

so we can apply Theorem 4.2 to obtain

$$\begin{aligned}
\mu\left((1-\lambda)X+\lambda Y\right) &= \int_{C\cap((1-\lambda)X+\lambda Y)} f(z)\,dz \\
&= \int_{\mathbb{R}^n} f(z)1_{C\cap((1-\lambda)X+\lambda Y)}(z)\,dz \\
&\geq \left(\int_{\mathbb{R}^n} f(x)1_{C\cap X}(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} f(x)1_{C\cap Y}(x)\,dx\right)^{\lambda} \\
&= \left(\int_{C\cap X} f(x)\,dx\right)^{1-\lambda}\left(\int_{C\cap Y} f(x)\,dx\right)^{\lambda} \\
&= \mu(X)^{1-\lambda}\mu(Y)^{\lambda}.
\end{aligned}$$

This observation has been generalized considerably, as follows. If $0<\lambda<1$ and $p\neq 0$, we define

$$M_p(a,b,\lambda)=((1-\lambda)a^p+\lambda b^p)^{1/p}$$

if $ab\neq 0$ and $M_p(a,b,\lambda)=0$ if $ab=0$; we also define

$$M_0(a,b,\lambda)=a^{1-\lambda}b^{\lambda},$$

$M_{-\infty}(a,b,\lambda)=\min\{a,b\}$, and $M_{\infty}(a,b,\lambda)=\max\{a,b\}$. These quantities and their natural generalizations for more than two numbers are called *pth means.* The classic text of Hardy, Littlewood, and Pólya [77] is still the best general reference. (Note, however, the different convention here when $p>0$ and $ab=0$.) *Jensen's inequality for means* (see [77, Section 2.9]) implies that if $-\infty\leq p<q\leq\infty$, then

$$M_p(a,b,\lambda)\leq M_q(a,b,\lambda), \tag{28}$$

with equality if and only if $a=b$ or $ab=0$.

A nonnegative function f on $\mathbb{R}^n$ is called *p-concave* on a convex set C if

$$f\left((1-\lambda)x+\lambda y\right)\geq M_p(f(x),f(y),\lambda),$$

for all $x,y\in C$ and $0<\lambda<1$. Analogously, we say that a finite (nonnegative) measure μ defined on (Lebesgue) measurable subsets of $\mathbb{R}^n$ is *p-concave* if

$$\mu\left((1-\lambda)X+\lambda Y\right)\geq M_p(\mu(X),\mu(Y),\lambda),$$

for all measurable sets X and Y in $\mathbb{R}^n$ and $0<\lambda<1$.

Thus 1-concave is just concave in the usual sense and 0-concave is log concave. The term *quasiconcave* is sometimes used for $-\infty$-concave. Also, if $p>0$ (or $p<0$), then f is p-concave if and only if f^p is concave (or convex, respectively). It follows from Jensen's inequality (28) that a p-concave function or measure is q-concave for all $q\leq p$.

Probability density functions of some important probability distributions are p-concave for some p. Consider, for example, the multivariate normal distribution on $\mathbb{R}^n$ with mean $m\in\mathbb{R}^n$ and $n\times n$ positive definite symmetric covariance matrix A. This has probability density

$$f(x)=c\exp\left(-\frac{(x-m)\cdot A^{-1}(x-m)}{2}\right),$$

where $c = (2\pi)^{-n/2}(\det A)^{-1/2}$. Since A is positive definite, the function $(x-m)\cdot A^{-1}(x-m)$ is convex and so f is log concave. The probability density functions of the Wishart, multivariate β, and Dirichlet distributions are also log concave; see [128]. The argument above then shows that the corresponding probability measures are log concave. Prékopa [128] explains how a problem from stochastic programming motivates this result.

However, Borell [28] noted that the density functions of the multivariate Pareto (the Cauchy distribution is a special case), t, and F distributions are not log concave, but are p-concave for some $p<0$. To obtain similar concavity conditions for the corresponding probability measures, a technical lemma is required.

Lemma 10.1. *Let $0<\lambda<1$ and let a, b, c, and d be nonnegative real numbers. If $p+q\ge 0$, then*

$$M_p(a,b,\lambda)M_q(c,d,\lambda) \ge M_s(ac,bd,\lambda),$$

where $s = pq/(p+q)$ if p and q are not both zero, and $s=0$ if $p=q=0$.

Proof. A general form of Hölder's inequality (see [77, p. 24]) states that when $0<\lambda<1$, $p_1,p_2,r>0$ with $1/p_1+1/p_2=1$, and a, b, c, and d are nonnegative real numbers, then

$$M_r(ac,bd,\lambda) \le M_{rp_1}(a,b,\lambda)M_{rp_2}(c,d,\lambda),$$

and that the inequality reverses when $r<0$. Suppose that $p+q>0$. If $p,q>0$, we can let $r=s$, $p_1=p/s$, and $p_2=q/s$, and the desired inequality follows immediately. If $p<0$, then $q>0$ and we let $r=p$, $p_1=s/p$, and $p_2=-q/p$; then replace a, b, c, and d, by ac, bd, $1/c$, and $1/d$, respectively. The remaining cases follow by continuity. □

The following theorem generalizes the Prékopa-Leindler inequality in $\mathbb{R}^n$, which is just the case $p=0$. The number $p/(np+1)$ is interpreted in the obvious way; it is equal to $-\infty$ when $p=-1/n$ and to $1/n$ when $p=\infty$.

Theorem 10.2. (Borell-Brascamp-Lieb inequality.) *Let $0<\lambda<1$, let $-1/n\le p\le\infty$, and let f, g, and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying*

$$h\left((1-\lambda)x+\lambda y\right) \ge M_p\left(f(x),g(y),\lambda\right),$$

for all $x,y\in\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,dx \ge M_{p/(np+1)}\left(\int_{\mathbb{R}^n} f(x)\,dx, \int_{\mathbb{R}^n} g(x)\,dx, \lambda\right).$$

Proof. This is very similar to the proof of the Prékopa-Leindler inequality. To deal with the case $n=1$, follow the second proof of Theorem 4.1, defining F, G, u, v, and w as in that theorem. Then, by Lemma 10.1 with $q=1$,

$$\begin{aligned}\int_{\mathbb{R}} h(x)\,dx &\ge \int_0^1 h\left(w(t)\right)w'(t)\,dt\\ &\ge \int_0^1 M_p\left(f\left(u(t)\right),g\left(v(t)\right),\lambda\right)M_1\left(\frac{F}{f\left(u(t)\right)},\frac{G}{g\left(v(t)\right)},\lambda\right)dt\\ &\ge \int_0^1 M_{p/(p+1)}(F,G,\lambda)\,dt = M_{p/(p+1)}(F,G,\lambda).\end{aligned}$$

The general case follows as in Theorem 4.2 by induction on n. □

Theorem 10.2 was proved (in slightly modified form) for $p > 0$ by Henstock and Macbeath [78] (when $n = 1$) and Dinghas [50]. The limiting case $p = 0$, as we noted above, was also proved by Prékopa and Leindler, and rediscovered by Brascamp and Lieb [33]. In general form Theorem 10.2 is stated and proved by Brascamp and Lieb [35, Theorem 3.3] and by Borell [28, Theorem 3.1] (but with a much more complicated proof; see also the paper of Rinott [130]). The proof above may be found in [44] and [47] (see also [49, Theorem 3.15]), but still draws on methods introduced by Henstock, Macbeath, and Dinghas. Das Gupta's survey [47] contains a very thorough examination and assessment of the various contributions and proofs before 1980. Brascamp and Lieb [35] obtain an "essential" form of Theorem 10.2, as in the case $p = 0$ (Theorem 9.1 above). Dancs and Uhrin [44] also offer a version of Theorem 10.2 for $-\infty \le p < -1/n$.

In calling Theorem 10.2 the *Borell-Brascamp-Lieb inequality* we are following the authors of [42] (who also generalize it to a Riemannian manifold setting; see Section 19.13) and placing the emphasis on the negative values of p. In fact, the proof of [42, Corollary 1.1] shows that the strongest inequality in this family is that for $p = -1/n$; that is, Theorem 10.2 for $p = -1/n$ implies Theorem 10.2 for all $p > -1/n$. This follows from a suitable rescaling of the functions f, g, and h, Lemma 10.1 with $q = -p/(np+1)$, and the observation that $M_p(a,b,\lambda)^{-1} = M_{-p}(1/a, 1/b, \lambda)$. The approach of Brascamp and Lieb [35], incidentally, was to observe that Theorem 10.2 also holds for $n = 1$ and $p = -\infty$ (the argument is contained in the first proof of Theorem 4.1), and then to derive Theorem 10.2 for $n = 1$ and $p \ge -1$ from this and Lemma 10.1.

Corollary 10.3. *Let $-1/n \le p \le \infty$ and let f be an integrable function that is p-concave on an open convex set C in $\mathbb{R}^n$ contained in its support. Then the measure generated by f and C is $p/(np+1)$-concave.*

Proof. This follows from Theorem 10.2 in exactly the same way as the special case $p = 0$ follows from the Prékopa-Leindler inequality (see the beginning of this section). □

The Brunn-Minkowski inequality says that Lebesgue measure in $\mathbb{R}^n$ is $1/n$-concave, and Theorem 10.2 supplies plenty of measures that are p-concave for $-1/n \le p \le \infty$. Borell [28] (see also [49, Theorem 3.17]) proves a sort of converse to Corollary 10.3: Given $-\infty \le p \le 1/n$ and a p-concave measure μ with n-dimensional support S, there is a $p/(1-np)$-concave function on S that generates μ. Borell also observed that when $p > 1/n$, no nontrivial p-concave measures exist in $\mathbb{R}^n$, and that any $1/n$-concave measure is a multiple of Lebesgue measure; see [49, Theorem 3.14]. Dancs and Uhrin [44, Theorem 3.4] find a generalization of Theorem 10.2 in which Lebesgue measure is replaced by a q-concave measure for some $-\infty \le q \le 1/n$.

It is convenient to mention here a sharpening of the Brunn-Minkowski theorem proved by Bonnesen in 1929 (see [44] and [134, p. 314]). If X is a bounded measurable set in $\mathbb{R}^n$, the *inner section function* m_X of X is defined by

$$m_X(u) = \sup_{t\in\mathbb{R}} V_{n-1}\left(X \cap (u^\perp + tu)\right),$$

for $u \in S^{n-1}$. (In 1926, Bonnesen asked if this function determines a convex body in $\mathbb{R}^n$, $n \ge 3$, up to translation and reflection in the origin, a question that remains unanswered; see [67, Problem 8.10].) Bonnesen proved that if $0 < \lambda < 1$ and $u \in S^{n-1}$, then

$$V_n\left((1-\lambda)X + \lambda Y\right) \geq M_{1/(n-1)}\left(m_X(u), m_Y(u), \lambda\right)\left((1-\lambda)\frac{V_n(X)}{m_X(u)} + \lambda\frac{V_n(Y)}{m_Y(u)}\right). \tag{29}$$

Lemma 10.1 with $p = 1/(n-1)$ and $q = 1$ shows that this is indeed stronger than (15). As Dancs and Uhrin [44, Theorem 3.2] show, an integral version of (29), in a general form similar to Theorem 10.2, can be constructed from the ideas already presented here.

At present the most general results in this direction are contained in the papers of Uhrin; see [146], [147], and the references given there. In particular, Uhrin states in [147, p. 306] that all previous results of this type are contained in [147, (3.42)]. The latter inequality has as an ingredient a "curvilinear Minkowski addition," and its proof reintroduces geometrical methods.

11. Convolutions

The *convolution* of measurable functions f and g on $\mathbb{R}^n$ is

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

The next two theorems, on concavity of products and sections of functions, are useful in obtaining a result on the concavity of convolutions.

Theorem 11.1. *Let $p_1 + p_2 \geq 0$, and let $p = p_1p_2/(p_1+p_2)$ if p_1 and p_2 are not both zero, and $p = 0$ if $p_1 = p_2 = 0$. For $i = 1, 2$, let f_i be a p_i-concave function on a convex set C_i in $\mathbb{R}^n$. Then the function $f(x,y) = f_1(x)f_2(y)$ is p-concave on $C_1 \times C_2$.*

Proof. Suppose that $0 < \lambda < 1$, and let $x_i \in C_1$ and $y_i \in C_2$ for $i = 0, 1$. By Lemma 10.1,

$$\begin{aligned} f\left((1-\lambda)(x_0,y_0) + \lambda(x_1,y_1)\right) &= f_1\left((1-\lambda)x_0 + \lambda x_1\right) f_2\left((1-\lambda)y_0 + \lambda y_1\right) \\ &\geq M_{p_1}\left(f_1(x_0), f_1(x_1), \lambda\right) M_{p_2}\left(f_2(y_0), f_2(y_1), \lambda\right) \\ &\geq M_p\left(f_1(x_0)f_2(y_0), f_1(x_1)f_2(y_1), \lambda\right) \\ &= M_p\left(f(x_0,y_0), f(x_1,y_1), \lambda\right). \end{aligned}$$

□

Theorem 11.2. *Let $p \geq -1/n$ and let f be an integrable p-concave function on an open convex set C in $\mathbb{R}^{m+n}$. For each x in the projection $C|\mathbb{R}^m$ of C onto $\mathbb{R}^m$, let $C(x) = \{y \in \mathbb{R}^n : (x,y) \in C\}$. Then*

$$F(x) = \int_{C(x)} f(x,y)\,dy$$

is $p/(np+1)$-concave on $C|\mathbb{R}^m$.

Proof. For $i = 0, 1$, let $x_i \in C|\mathbb{R}^m$ and let $g_i(y) = f(x_i, y)$ for $y \in C(x_i)$. Suppose that $0 < \lambda < 1$ and that $x = (1-\lambda)x_0 + \lambda x_1$, and let $g(y) = f(x,y)$ for $y \in C(x)$. The p-concavity of f implies that

$$g\left((1-\lambda)y_0 + \lambda y_1\right) \geq M_p\left(g_0(y_0), g_1(y_1), \lambda\right)$$

whenever $y_i \in C(x_i)$, $i = 0, 1$. Also,

$$C(x) \supset (1-\lambda)C(x_0) + \lambda C(x_1).$$

Then Theorem 10.2 yields

$$\int_{C(x)} g(y)\,dy \geq M_{p/(np+1)}\left(\int_{C(x_0)} g_0(y)\,dy, \int_{C(x_1)} g_1(y)\,dy, \lambda\right).$$

This shows that F is $p/(np+1)$-concave on $C|\mathbb{R}^m$. □

If we apply the previous theorem with $n = 1$ and $f = 1_C$ when C is the interior of a convex body K in $\mathbb{R}^{m+1}$, and let $p \to \infty$, we see that the function giving volumes of parallel hyperplane sections of K is $1/n$-concave. This statement is equivalent to the Brunn-Minkowski inequality for convex bodies.

Theorem 11.3. *Let $p_1 + p_2 \geq 0$, and let $p = p_1p_2/(p_1 + p_2)$ if p_1 and p_2 are not both zero, and $p = 0$ if $p_1 = p_2 = 0$. Suppose further that $p \geq -1/n$. For $i = 1, 2$, let f_i be an integrable p_i-concave function on an open convex set C_i in $\mathbb{R}^n$. Then $f_1 * f_2$ is $p/(np+1)$-concave on C_1+C_2.*

Proof. By Theorem 11.1, the function $f_1(x - y)f_2(y)$ is p-concave for $(x - y, y) \in C_1 \times C_2 \subset \mathbb{R}^{2n}$, that is, for $x \in C_1 + C_2$. The result follows from Theorem 11.2. □

For extensions to measures and some examples that limit the possibility of weakening the conditions on p_1, p_2, and p in Theorem 11.3, see [49, Section 3.3], whose general approach we have followed in this section. Theorem 11.2 can be found in [28] and [35]. The early history of Theorem 11.3 (when $p = 0$, this says that the convolution of two log concave functions is also log concave) is discussed by Das Gupta [47, p. 313].

12. The covariogram

Theorem 12.1. *Let K and L be convex bodies in $\mathbb{R}^n$. Then the function*

$$g_{K,L}(x) = V\left(K \cap (L + x)\right)^{1/n},$$

for $x \in \mathbb{R}^n$, is concave on its support.

Proof. For $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$, we have

$$\begin{aligned} K \cap (L + (1-\lambda)x + \lambda y) &= K \cap ((1-\lambda)(L + x) + \lambda(L + y)) \\ &\supset (1-\lambda)\left(K \cap (L + x)\right) + \lambda\left(K \cap (L + y)\right). \end{aligned}$$

Using the Brunn-Minkowski inequality (15), we obtain

$$\begin{aligned} g_{K,L}\left((1-\lambda)x + \lambda y\right) &\geq V\left((1-\lambda)\left(K \cap (L + x)\right) + \lambda\left(K \cap (L + y)\right)\right)^{1/n} \\ &\geq (1-\lambda)V\left(K \cap (L + x)\right)^{1/n} + \lambda V\left(K \cap (L + y)\right)^{1/n} \\ &= (1-\lambda)g_{K,L}(x) + \lambda g_{K,L}(y), \end{aligned}$$

as required. □

As a corollary, we conclude that the *covariogram* g_K of a convex body K in $\mathbb{R}^n$, defined for $x \in \mathbb{R}^n$ by

$$g_K(x) = V\left(K \cap (K + x)\right),$$

is $1/n$-concave (and hence log concave) on its support, which, it is easy to check, is the *difference body* $DK = K + (-K)$ of K. Obviously g_K is unchanged when K is translated or replaced by its reflection $-K$ in the origin. Note that

$$\begin{aligned} g_K(x) &= \int_{\mathbb{R}^n} 1_{K\cap(K+x)}(y)\,dy \\ &= \int_{\mathbb{R}^n} 1_K(y)1_{K+x}(y)\,dy \\ &= \int_{\mathbb{R}^n} 1_K(y)1_K(y-x)\,dy = 1_{-K} * 1_K(x). \end{aligned}$$

The name "covariogram" stems from the theory of random sets, where the covariance is defined for $x \in \mathbb{R}^n$ as the probability that both o and x lie in the random set. The covariogram is also useful in mathematical morphology. See [135, Chapter 9]) and [140, Section 6.2]. In 1986, G. Mathéron (see the references in [132]) asked if the covariogram determines convex bodies, up to translation and reflection in the origin. Remarkably, this question is open even for $n = 2$! Nagel [121] proved that the answer is affirmative when K and L are convex polygons in the plane. Bianchi [23] has shown that the answer is affirmative for much larger class of planar convex bodies. He has also found pairs of convex polyhedra that represent counterexamples in $\mathbb{R}^4$, but these are still reflections of each other in a plane. See also [70, Section 6], and the references given in connection with chord-power integrals in [67, p. 267].

13. Anderson's theorem

Anderson [2] used the Brunn-Minkowski theorem in his work on multivariate unimodality. He began with the following simple observation. If f is a (i) symmetric ($f(x) = f(-x)$) and (ii) unimodal ($f(cx) \geq f(x)$ for $0 \leq c \leq 1$) function on $\mathbb{R}$, and I is an interval centered at the origin, then

$$\int_{I+y} f(x)\,dx$$

is maximized when $y = 0$. In probability language, if a random variable X has probability density f and Y is an independent random variable, then

$$\mathrm{Prob}\,\{X \in I\} \geq \mathrm{Prob}\,\{X + Y \in I\}.$$

To see this, recall that if g is the probability density of Y, then $f*g$ is the probability density of $X+Y$; see [82, Section 11.5]. So, by Fubini's theorem,

$$\begin{aligned}
\mathrm{Prob}\,\{X+Y\in I\} &= \int_I\int_{\mathbb{R}} f(z-y)g(y)\,dy\,dz\\
&= \int_{\mathbb{R}}\int_I f(z-y)g(y)\,dz\,dy\\
&= \int_{\mathbb{R}}\int_{I-y} f(x)g(y)\,dx\,dy\\
&\le \int_{\mathbb{R}}\int_I f(x)g(y)\,dx\,dy\\
&= \int_I f(x)\,dx = \mathrm{Prob}\,\{X\in I\}.
\end{aligned}$$

Anderson generalized this, as follows. If f is a nonnegative function on $\mathbb{R}^n$, call f *unimodal* if the level sets $L(f,t)$ (see (4)) are convex for each $t\ge 0$. Note that every quasiconcave function and hence all p-concave functions are unimodal.

Theorem 13.1. (Anderson's theorem.) *Let K be an origin-symmetric (i.e., $K=-K$) convex body in $\mathbb{R}^n$ and let f be a nonnegative, symmetric, and unimodal function integrable on $\mathbb{R}^n$. Then*

$$\int_K f(x+cy)\,dx \ge \int_K f(x+y)\,dx,$$

for $0\le c\le 1$ and $y\in\mathbb{R}^n$.

Proof. Suppose initially that $f(x)=1_L(x)$, where L is an origin-symmetric convex body in $\mathbb{R}^n$. Then $f(x+y)=1_L(x+y)=1_{L-y}(x)$ and

$$\int_K f(x+y)\,dx = \int_K 1_{L-y}(x)\,dx = V\left(K\cap(L-y)\right) = g_{K,L}(-y) = g_{K,L}(y).$$

Theorem 12.1 implies that $g_{K,L}$ is log concave. Let $\lambda=(1-c)/2$. Since

$$\begin{aligned}
g_{K,L}(cy) &= g_{K,L}\left((1-2\lambda)y\right)\\
&= g_{K,L}\left((1-\lambda)y+\lambda(-y)\right)\\
&\ge g_{K,L}(y)^{1-\lambda}g_{K,L}(-y)^{\lambda}\\
&= g_{K,L}(y)^{1-\lambda}g_{K,L}(y)^{\lambda} = g_{K,L}(y),
\end{aligned}$$

the theorem follows. In the general case, $L(f,t)$ is an origin-symmetric convex body, so by (6), Fubini's theorem, and the special case just proved,

$$\begin{aligned}\int_K f(x+cy)\,dx &= \int_K\int_0^\infty 1_{L(f,t)}(x+cy)\,dt\,dx\\ &= \int_0^\infty\int_K 1_{L(f,t)}(x+cy)\,dx\,dt\\ &\ge \int_0^\infty\int_K 1_{L(f,t)}(x+y)\,dx\,dt\\ &= \int_K f(x+y)\,dx.\end{aligned}$$

□

Anderson's theorem says that the integral of a symmetric unimodal function f over an n-dimensional centrally symmetric convex body K does not decrease when K is translated towards the origin. Since the graph of f forms a hill whose peak is over the origin, this is intuitively clear. However, it is no longer obvious, as it was in the 1-dimensional case! There may be points $x \in K$ at which the value of f is larger than it is at the corresponding translate of x.

As above, we can conclude from Anderson's theorem that if a random variable X has probability density f on $\mathbb{R}^n$ and Y is an independent random variable, then

$$\mathrm{Prob}\,\{X \in K\} \ge \mathrm{Prob}\,\{X+Y \in K\},$$

where K is any origin-symmetric convex body in $\mathbb{R}^n$. We noted above that density functions of some well-known probability distributions are p-concave for some p, and hence unimodal. If they are also symmetric, Anderson's theorem applies.

Suppose K is a convex body in $\mathbb{R}^n$, $y \in \mathbb{R}^n$, $p \ge -1/n$, and f is an integrable p-concave function on $\mathbb{R}^n$. Corollary 10.3 implies that the measure μ generated by f and $\mathbb{R}^n$ is $p/(np+1)$-concave on $\mathbb{R}^n$. Let

$$h(y) = \mu(K-y) = \int_{K-y} f(x)\,dx = \int_K f(x+y)\,dx.$$

Since

$$K-(1-\lambda)y_0-\lambda y_1 = (1-\lambda)(K-y_0)+\lambda(K-y_1),$$

we have

$$\begin{aligned}h\left((1-\lambda)y_0+\lambda y_1\right) &= \mu\left(K-(1-\lambda)y_0-\lambda y_1\right)\\ &= \mu\left((1-\lambda)(K-y_0)+\lambda(K-y_1)\right)\\ &\ge M_{p/(np+1)}\left(\mu(K-y_0),\mu(K-y_1),\lambda\right)\\ &= M_{p/(np+1)}\left(h(y_0),h(y_1),\lambda\right).\end{aligned}$$

Therefore h is $p/(np+1)$-concave on $\mathbb{R}^n$ and hence unimodal. In particular, $h(cy)$ is unimodal in c for a fixed y. This shows that Corollary 10.3 and Anderson's theorem are related. Anderson's theorem replaces the restriction $p \ge -1/n$ with a much weaker condition, but requires in exchange the symmetry of f and K.

Anderson's theorem has many applications in probability and statistics, where, for example, it can be applied to show that certain statistical tests are unbiased. See [2], [36], [49], and [144].

14. Young's inequality

We saw in the previous sections how the Brunn-Minkowski inequality and convolutions come together naturally. The next theorem provides two convolution inequalities with sharp constants, the first proved independently by Beckner [21] and Brascamp and Lieb [34], and the second by Brascamp and Lieb [34]. (See Section 17 for more information.) We shall soon see that the second inequality actually implies the Brunn-Minkowski inequality.

Theorem 14.1. *Let $0 < p, q, r$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{30}$$

and let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be nonnegative. Then

$$\text{(Young's inequality)} \qquad \|f * g\|_r \le C^n \|f\|_p \|g\|_q, \qquad \text{for } p, q, r \ge 1, \tag{31}$$

and

$$\text{(Reverse Young inequality)} \qquad \|f * g\|_r \ge C^n \|f\|_p \|g\|_q, \qquad \text{for } p, q, r \le 1. \tag{32}$$

Here $C = C_p C_q / C_r$, where

$$C_s^2 = \frac{|s|^{1/s}}{|s'|^{1/s'}} \tag{33}$$

for $1/s + 1/s' = 1$ (that is, s and s' are Hölder conjugates).

The inequality (31), when expanded, reads as follows:

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) g(y)\, dy \right)^r dx \right)^{1/r} \le C^n \left(\int_{\mathbb{R}^n} f(x)^p\, dx \right)^{1/p} \left(\int_{\mathbb{R}^n} g(x)^q\, dx \right)^{1/q}.$$

Inequalities (31) and (32) together show that equality holds in both when $p = q = r = 1$. In fact, since $C_p \to 1$ as $p \to 1$, when $p = q = r = 1$ we have $C = 1$, and substituting $u = x - y$, $v = y$ in the left-hand side of (31), we obtain

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u) g(v)\, dv\, du \le \int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} g(x)\, dx.$$

But equality holds here and therefore also in (31), and similarly in (32).

Theorem 14.2. *The limiting case $r \to 0$ of the reverse Young inequality is the essential form of the Prékopa-Leindler inequality in $\mathbb{R}^n$ (Theorem 9.1).*

Proof. Let f_m and g_m be sequences of bounded measurable functions with compact support converging in $L^1(\mathbb{R}^n)$ to f and g, respectively, as $m \to \infty$ and satisfying $f_m \le f$ and $g_m \le g$. Let

$$s_m(x) = \operatorname*{ess\,sup}_y f_m \left(\frac{x-y}{1-\lambda} \right)^{1-\lambda} g_m \left(\frac{y}{\lambda} \right)^{\lambda}. \tag{34}$$

Let $s(x)$ be defined by replacing f_m by f and g_m by g in (34). As in the proof of Theorem 9.1, s and each s_m is measurable. Also, $\|s\|_1 \ge \|s_m\|_1$, so if

$$\|s_m\|_1 \ge \|f_m\|_1^{1-\lambda} \|g_m\|_1^{\lambda}$$

for each m we have

$$\|s\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

Therefore it suffices to prove the theorem when f and g are bounded measurable functions with compact support.

Assuming this, note that $s(x) = \lim_{m\to\infty} S_m(x)$, where

$$S_m(x) = \left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{(1-\lambda)m} g\left(\frac{y}{\lambda}\right)^{\lambda m} dy\right)^{1/(m-1)}.$$

(If we replaced the exponent, $1/(m-1)$ by $1/m$, this would follow from the fact that the mth mean tends to the supremum as $m \to \infty$; compare [77, p. 143]. But this replacement is irrelevant in the limit.) Note also that $\|s\|_1 = \lim_{m\to\infty}\|S_m\|_1$ (we can interchange the limit and integral because the S_m's are uniformly bounded and have supports lying in some common compact set).

Applying the reverse Young inequality to S_m with $m > \max\{(1-\lambda)^{-1}, \lambda^{-1}\}$,
$p = 1/((1-\lambda)m)$, $q = 1/(\lambda m)$, and $r = 1/(m-1)$, we obtain

$$\begin{aligned}
\|S_m\|_1 &= \int_{\mathbb{R}^n} S_m(x)\,dx \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{(1-\lambda)m} g\left(\frac{y}{\lambda}\right)^{\lambda m} dy\right)^{1/(m-1)} dx \\
&\geq \left(C^n\left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right) dx\right)^{(1-\lambda)m}\left(\int_{\mathbb{R}^n} g\left(\frac{y}{\lambda}\right) dy\right)^{\lambda m}\right)^{1/(m-1)} \\
&= C^{n/(m-1)}\left((1-\lambda)^n\|f\|_1\right)^{(1-\lambda)m/(m-1)}\left(\lambda^n\|g\|_1\right)^{\lambda m/(m-1)}.
\end{aligned}$$

Therefore

$$\|s\|_1 = \lim_{m\to\infty}\|S_m\|_1 \geq \left((1-\lambda)^{1-\lambda}\lambda^{\lambda}\lim_{m\to\infty} C^{1/(m-1)}\right)^n \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

It remains only to check that

$$\lim_{m\to\infty} C^{1/(m-1)} = (1-\lambda)^{-(1-\lambda)}\lambda^{-\lambda}.$$

□

15. The Brascamp-Lieb inequality and Barthe's inequality

The inequalities presented in this section approach the most general known in the direction of Young's inequality and its reverse form, and represent a research frontier that can be expected to move before too long.

Each $m \times n$ matrix A defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and this linear map can also be denoted by A. The Euclidean adjoint A^* of A is then an $n \times m$ matrix or linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ satisfying $Ax \cdot y = x \cdot A^*y$ for each $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

Theorem 15.1. *Let $c_i > 0$ and $n_i \in \mathbb{N}$, $i = 1, \ldots, m$, with $\sum_i c_i n_i = n$. Let $f_i \in L^1(\mathbb{R}^{n_i})$ be nonnegative and let $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ be a linear surjection, $i = 1, \ldots, m$. Then*

(Brascamp-Lieb inequality)

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(B_i x)^{c_i}\,dx \le D^{-1/2} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x)\,dx \right)^{c_i} \tag{35}$$

and
(Barthe's inequality)

$$\overline{\int}_{\mathbb{R}^n} \sup\left\{ \prod_{i=1}^m f_i(z_i)^{c_i} : x = \sum_i c_i B_i^* z_i,\ z_i \in \mathbb{R}^{n_i} \right\} dx \ge D^{1/2} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x)\,dx \right)^{c_i}, \tag{36}$$

where

$$D = \inf\left\{ \frac{\det\left(\sum_{i=1}^m c_i B_i^* A_i B_i\right)}{\prod_{i=1}^m (\det A_i)^{c_i}} : A_i \text{ is a positive definite } n_i \times n_i \text{ matrix} \right\}. \tag{37}$$

For comments on equality conditions and ideas of proof, including a proof of an important special case of (36), see Section 17.

We can begin to understand (35) by taking $n_i = n$, $B_i = I_n$, the identity map on $\mathbb{R}^n$, replacing f_i by f_i^{1/c_i}, and letting $c_i = 1/p_i$, $i = 1, \dots, m$. Then $\sum_i 1/p_i = 1$ and the log concavity of the determinant of a positive definite matrix (see, for example, [20, p. 63]) yields $D = 1$. Therefore

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x)\,dx \le \prod_{i=1}^m \|f_i\|_{p_i},$$

Hölder's inequality in $\mathbb{R}^n$.

Next, take $m = 2$, $n_1 = n_2 = n$, $B_1 = B_2 = I_n$, $c_1 = 1 - \lambda$, and $c_2 = \lambda$ in (36). Again we have $D = 1$, so

$$\overline{\int}_{\mathbb{R}^n} \sup\left\{ f_1(z_1)^{1-\lambda} f_2(z_2)^{\lambda} :\ x = (1-\lambda)z_1 + \lambda z_2 \right\} dx \ge \left(\int_{\mathbb{R}^n} f_1(x)\,dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f_2(x)\,dx \right)^{\lambda},$$

the Prékopa-Leindler inequality (12) in $\mathbb{R}^n$.

Theorem 15.2. (Young's inequality in $\mathbb{R}^n$, second form.) *Let $0 < p, q, r$ satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2,$$

and let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, and $h \in L^r(\mathbb{R}^n)$ be nonnegative. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(x-y) h(y)\,dy\,dx \le \overline{C}^n \|f\|_p \|g\|_q \|h\|_r, \tag{38}$$

where $\overline{C} = C_p C_q C_r$ is defined using (33).

Theorem 15.3. *The second form of Young's inequality in $\mathbb{R}^n$ is equivalent to the first (31).*

Proof. Let $p, q, r \geq 1$ satisfy (30). By Hölder's inequality (11),

$$\begin{aligned}
&\sup\left\{\frac{\|f*g\|_r}{\|f\|_p\|g\|_q} : f\in L^p(\mathbb{R}^n), g\in L^q(\mathbb{R}^n)\right\} = \\
&= \sup\left\{\frac{\int_{\mathbb{R}^n}(f*g)(x)h(x)\,dx}{\|f\|_p\|g\|_q\|h\|_{r'}} : f\in L^p(\mathbb{R}^n), g\in L^q(\mathbb{R}^n), h\in L^{r'}(\mathbb{R}^n)\right\}\\
&= \sup\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x-y)g(y)h(x)\,dx\,dy}{\|f\|_p\|g\|_q\|h\|_{r'}} : f\in L^p(\mathbb{R}^n), g\in L^q(\mathbb{R}^n), h\in L^{r'}(\mathbb{R}^n)\right\}\\
&= \sup\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x)g(x-y)h(y)\,dy\,dx}{\|f\|_{\overline{p}}\|g\|_{\overline{q}}\|h\|_{\overline{r}}} : f\in L^{\overline{p}}(\mathbb{R}^n), g\in L^{\overline{q}}(\mathbb{R}^n), h\in L^{\overline{r}}(\mathbb{R}^n)\right\},
\end{aligned}$$

where the last equality is obtained by replacing f, g, h, p, q, and r', by g, h, f, $\overline{q}$, $\overline{r}$, and $\overline{p}$, respectively, so that

$$\frac{1}{\overline{p}}+\frac{1}{\overline{q}}+\frac{1}{\overline{r}}=2.$$

□

Theorem 15.4. *The Brascamp-Lieb inequality (35) implies Young's inequality in* $\mathbb{R}^n$.

Proof. In (35), let $m = 3$, $n_1 = n_2 = n_3 = n$, and let $B_i : \mathbb{R}^{2n} \to \mathbb{R}^n$, $i = 1, 2, 3$ be the linear maps taking $(z_1, \dots, z_{2n})$ to $(z_1, \dots, z_n)$, $(z_1 - z_{n+1}, \dots, z_n - z_{2n})$, and $(z_{n+1}, \dots, z_{2n})$, respectively; then replace f_i by f_i^{1/c_i}, $i = 1, 2, 3$ and let $c_1 = 1/p$, $c_2 = 1/q$, and $c_3 = 1/r$. In this case $D = C^{-2}$, where C is as in Theorem 14.1; see [34, Theorem 5]. This gives

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(x)f_2(x-y)f_3(y)\,dy\,dx \leq C\|f_1\|_p\|f_2\|_q\|f_3\|_r,$$

which is (38). □

As a side remark, we note that there is a version of Young's inequality in its second form (38), called the *weak Young inequality*, which only requires that $g \in L^q_w(\mathbb{R}^n)$, the weak L^q space. See [91, Section 4.3] for details. This allows one to conclude in particular that under the (slightly weakened) hypotheses of Theorem 15.2, with $q = n/\lambda$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\|x-y\|^{-\lambda}h(y)\,dy\,dx \leq k(n,\lambda,p)\|f\|_p\|h\|_r. \tag{39}$$

This was proved in Lieb [89] with a sharp constant $k(n, \lambda, p)$. The classical form without the sharp constant is called the *Hardy-Littlewood-Sobolev* inequality. The case $\lambda = n - 2$ is of particular interest in potential theory, as is explained in [91, Chapter 9].

16. Back to geometry

As Ball [13] remarks, some geometry comes back into view if we replace $f(x)$ in Young's inequality (38) by $f(-x)$:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(-x_1)g(x_1-x_2)h(x_2)\,dx_2\,dx_1 \leq \overline{C}\|f\|_p\|g\|_q\|h\|_r. \tag{40}$$

Define $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\phi(x_1, x_2) = z = (z_1, z_2, z_3)$, where $z_1 = -x_1$, $z_2 = x_1 - x_2$, and $z_3 = x_2$. Then $\phi(\mathbb{R}^2) = S$, where S is the plane $\{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0\}$ through the origin. Let $f = g = h = 1_{[-1,1]}$ and $C_0 = [-1, 1]^3$. By (40),

$$\begin{aligned} V_2(C_0 \cap S) &= \int_S 1_{C_0}(z)\, dz \\ &= \int_S f(z_1)g(z_2)h(z_3)\, dz \\ &= J(\phi)^{-1} \int_{\mathbb{R}}\int_{\mathbb{R}} f(-x_1)g(x_1 - x_2)h(x_2)\, dx_2\, dx_1, \end{aligned}$$

where $J(\phi)$ is the Jacobian of ϕ. So Young's inequality might be used to provide upper bounds for volumes of central sections of cubes. In fact, Ball [9] used the following special case of the Brascamp-Lieb inequality to do just this.

Suppose that $c_i > 0$ and $u_i \in S^{n-1}$, $i = 1, \dots, m$ satisfy

$$x = \sum_{i=1}^{m} c_i (x \cdot u_i) u_i,$$

for all $x \in \mathbb{R}^n$. This says that the u_i's are acting like an orthonormal basis for $\mathbb{R}^n$. The condition is often written

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{41}$$

where $u \otimes u$ denotes the rank one orthogonal projection onto the span of u, the map that sends x to $(x \cdot u)u$. Taking traces in (41), we see that

$$\sum_{i=1}^{m} c_i = n. \tag{42}$$

Theorem 16.1. *Let $c_i > 0$ and $u_i \in S^{n-1}$, $i = 1, \dots, m$ be such that (41) and hence (42) holds. If $f_i \in L^1(\mathbb{R})$ is nonnegative, $i = 1, \dots, m$, then*
(Geometric Brascamp-Lieb inequality)

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x \cdot u_i)^{c_i}\, dx \le \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i(x)\, dx \right)^{c_i} \tag{43}$$

and
(Geometric Barthe inequality)

$$\overline{\int}_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i(z_i)^{c_i} : x = \sum_i c_i z_i u_i,\ z_i \in \mathbb{R} \right\} dx \ge \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i(x)\, dx \right)^{c_i}. \tag{44}$$

Proof. Let $n_i = 1$ and for $x \in \mathbb{R}^n$, let $B_i x = x \cdot u_i$, $i = 1, \dots, m$. Then $B_i^* z_i = z_i u_i \in \mathbb{R}^n$ for $z_i \in \mathbb{R}$. The inequalities (35) and (36) become (43) and (44), respectively, because the hypotheses of the theorem and (37) imply that $D = 1$ (see [17, Proposition 9] for the details). $\square$

Note that the geometric Barthe inequality (44) still implies the Prékopa-Leindler inequality in $\mathbb{R}$, with the geometric consequences explained above.

Ball [9] used (43) to obtain the best-possible upper bound

$$V_k(C_0 \cap S) \le (\sqrt{2})^{n-k}$$

for sections of the cube $C_0 = [-1,1]^n$ by k-dimensional subspaces S, $1 \le k \le n-1$, when $2k \ge n$. (For smaller values of k, the best-possible bound is not known except for some special cases; see [9].) He also showed that (43) provides best-possible upper bounds for the *volume ratio* $vr(K)$ of a convex body K in $\mathbb{R}^n$, defined by

$$vr(K) = \left(\frac{V(K)}{V(E)}\right)^{1/n},$$

where E is the ellipsoid of maximal volume contained in K. The ellipsoid E is called the *John ellipsoid* of K. The following theorem is a refinement of Ball [12] of a theorem proved by Fritz John.

Theorem 16.2. *The John ellipsoid of a convex body K in $\mathbb{R}^n$ is B if and only if $B \subset K$ and there is an $m \ge n$, $c_i > 0$ and $u_i \in S^{n-1} \cap \partial K$, $i = 1, \dots, m$ such that (41) holds and $\sum_i c_i u_i = o$.*

Ball's argument is as follows. Let K be a convex body in $\mathbb{R}^n$. Since $vr(K)$ is affine invariant, we may assume that the John ellipsoid of K is B. If we can show that $V(K) \le 2^n$, then $vr(K) \le vr(C_0)$, where $C_0 = [-1,1]^n$. Let c_i and u_i be as in John's theorem, and note that the points u_i are *contact points*, points where the boundaries of K and B meet. If K is origin-symmetric and u_i is a contact point, then so is $-u_i$; therefore $K \subset L$, where

$$L = \{x \in \mathbb{R}^n : |x \cdot u_i| \le 1, i = 1, \dots, m\}$$

is the closed slab bounded by the hyperplanes $\{x : x \cdot u_i = \pm 1\}$. Also, if $f_i = 1_{[-1,1]}$, then

$$1_L(x) = \prod_{i=1}^m f_i(x \cdot u_i)^{c_i}.$$

By (43) and (42),

$$\begin{aligned} V(K) \le V(L) &= \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x \cdot u_i)^{c_i}\,dx \\ &\le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i(x)\,dx\right)^{c_i} = \prod_{i=1}^m 2^{c_i} = 2^n. \end{aligned}$$

This argument shows that $vr(K)$ is maximal for centrally symmetric K when K is a parallelotope.

One consequence of this estimate is the following result of Ball [11] (Behrend [22] proved the result for $n = 2$).

Theorem 16.3. (Reverse isoperimetric inequality for centrally symmetric convex bodies in $\mathbb{R}^n$.) *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ and let $C_0 = [-1,1]^n$. There is an affine* *transformation ϕ such that*

$$\left(\frac{S(\phi K)}{S(C_0)}\right)^{1/(n-1)} \le \left(\frac{V(\phi K)}{V(C_0)}\right)^{1/n}. \tag{45}$$

Proof. Choose ϕ so that the John ellipsoid of ϕK is B. The above argument shows that $V(\phi K) \le 2^n$. Since $B \subset \phi K$, we have, by (3),

$$\begin{aligned} S(\phi K) &= \lim_{\varepsilon\to 0+} \frac{V(\phi K + \varepsilon B) - V(\phi K)}{\varepsilon} \\ &\le \lim_{\varepsilon\to 0+} \frac{V(\phi K + \varepsilon \phi K) - V(\phi K)}{\varepsilon} \\ &= V(\phi K) \lim_{\varepsilon\to 0+} \frac{(1+\varepsilon)^n - 1}{\varepsilon} \\ &= nV(\phi K) = nV(\phi K)^{(n-1)/n} V(\phi K)^{1/n} \le 2nV(\phi K)^{(n-1)/n}. \end{aligned}$$

This is equivalent to (45). □

Of course, one cannot expect a reverse isoperimetric inequality without use of an affine transformation, since we can find convex bodies of any prescribed volume that are very flat and so have large surface area.

In [11], Ball used the same methods to show that for arbitrary convex bodies, the volume ratio is maximal for simplices, and to obtain a corresponding reverse isoperimetric inequality. The fact that the volume ratio is *only* maximal for parallelotopes (in the centrally symmetric case) or simplices was shown by Barthe [17] as a corollary of his study of the equality conditions in the Brascamp-Lieb inequality.

For other results of this type that employ Theorem 16.1, see [10], [16], and [133]. Barthe [17] states a multidimensional generalization of Theorem 16.1, also derived from Theorem 15.1, that leads to a multidimensional Brunn-Minkowski-type theorem.

17. More on history, proofs, and equality conditions

The *classical Young inequality* is

$$\|f * g\|_r \le \|f\|_p \|g\|_q, \qquad \text{for } p,q,r \ge 1,$$

that is, (31) with the better constant C^n there replaced by 1, under the same assumptions. This can be proved in a few lines using Hölder's inequality (11); see [91, p. 99]. It was proved by W. H. Young in 1912–13 (see [77, Sections 8.3 and 8.4] and the references given there), and is related to the *classical Hausdorff-Young inequality*: If $1 \le p \le 2$ and $f \in L^p(\mathbb{R}^n)$, then

$$\|\hat{f}\|_{p'} \le \|f\|_p, \tag{46}$$

where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{2\pi i x\cdot y}\, dy$$

of f, and p and p' are Hölder conjugates. This was proved by Hausdorff and Young for Fourier series, and extended to integrals by Titchmarsh in 1924. Beckner [21], improving earlier partial results of Babenko, showed that when $1 \le p \le 2$,

$$\|\hat{f}\|_{p'} \le C_p^n \|f\|_p, \tag{47}$$

where C_p is given by (33). (Lieb [90] proved that equality holds only for Gaussians.) This improvement on (46) is related to Young's inequality (31); in fact, the classical Young inequality was motivated by (46). To see the connection, suppose that (47) holds, $n = 1$, and $1 \le p, q, r' \le 2$. If p, q, r satisfy (30), then their Hölder conjugates satisfy $1/p' + 1/q' = 1/r'$. Using this and Hölder's inequality (11), we obtain

$$\begin{aligned} \|f * g\|_r &\le C_{r'} \|\hat{f}\hat{g}\|_{r'} \\ &\le C_{r'} \|\hat{f}\|_{p'} \|\hat{g}\|_{q'} \\ &\le C_{r'} (C_p \|f\|_p)(C_q \|g\|_q) = C \|f\|_p \|g\|_q. \end{aligned}$$

A similarly easy argument (see [21, pp.169–70]) shows that Young's inequality (31) yields (46) when p' is an even integer.

Young's inequality in the sharp form (31) was proved independently by Beckner [21] and Brascamp and Lieb [34]. The reverse Young inequality without the sharp constant (that is, with C replaced by 1) is due to Leindler [87]; the sharp version was obtained by Brascamp and Lieb [34]. The latter also found the connection to the Prékopa-Leindler inequality, Theorem 14.2, and established the following equality conditions: *When $n = 1$ and $p, q \ne 1$, equality holds in (31) or (32) if and only if f and g are Gaussians:*

$$f(x) = ae^{-c|p'|(x-\alpha)^2}, \; g(x) = be^{-c|q'|(x-\beta)^2},$$

for some a, b, c, α, β with $a, b \ge 0$ and $c > 0$.

The simplest known proof of Young's inequality and its reverse form, with the above equality conditions, was found by Barthe [18].

The Brascamp-Lieb inequality in the general form (35), with equality conditions, was proved by Lieb [90]. The special case $n_i = 1$ and $B_i x = x \cdot v_i$, where $x \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$, $i = 1, \dots, m$ is the main result of Brascamp and Lieb [34].

Let A be an $n \times n$ positive definite symmetric matrix, and let

$$G_A(x) = \exp(-Ax \cdot x),$$

for $x \in \mathbb{R}^n$. The function G_A is called a *centered Gaussian.* Lieb [90] proved that the supremum of the left-hand side of (35) for functions f_i of norm one is the same as the supremum of the left-hand side of (35) for centered Gaussians of norm one; in other words, the constant D can be computed using centered Gaussians.

There is also a version of (35) in which a fixed centered Gaussian appears in the integral on the left-hand side and the constant is again determined by taking the functions f_i to be Gaussians; see [34, Theorem 6], where an application to statistical mechanics is given, and [90, Theorem 6.2].

Barthe [17] proved (36), giving at the same time a simpler approach to (35) and its equality conditions.

The fact that the constant D in the geometric Brascamp-Lieb inequality (43) becomes 1 was observed by Ball [9]. Inequality (44) was first proved by Barthe [14]. As in the general case, equality holds in (43) and (44) for centered Gaussians.

The main idea behind Barthe's approach is the use of a familiar construction from measure theory. Let μ be a finite Borel measure in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ a Borel-measurable map defined μ-almost everywhere. For Borel sets M in $\mathbb{R}^n$, let

$$\nu(M) = (T\mu)(M) = \mu(T^{-1}(M)).$$

The Borel measure $\nu = T\mu$ is sometimes called the *push-forward* of μ by T, and T is said to *push forward* or *transport* the measure μ to ν. Suppose for simplicity that μ and ν are absolutely continuous with respect to Lebesgue measure, so that

$$\mu(M) = \int_M f(x)\,dx \text{ and } \nu(M) = \int_M g(x)\,dx$$

for Borel sets M in $\mathbb{R}^n$, and T is a differentiable bijection. Then

$$f(x) = g(T(x))J(T)(x),$$

where $J(T)$ is the Jacobian of T, and we can talk of T transporting f to g. If μ and ν are measures on $\mathbb{R}$, absolutely continuous with respect to Lebesgue measure and with $\mu(\mathbb{R}) = \nu(\mathbb{R})$, then we can always find a T that transports μ to ν, by defining $T(t)$ to be the smallest number such that

$$\int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{T(t)} g(x)\,dx.$$

Moreover, if f and g are continuous and positive, then T is strictly increasing and C^1, and

$$f(x) = g(T(x))T'(x).$$

In fact, the same parametrization was used in proving the Prékopa-Leindler inequality in $\mathbb{R}$. To see this, replace the functions f and g in the second proof of Theorem 4.1 with g_1 and g_2, respectively. If $f_i = F_i 1_{[0,1]}$, $i = 1, 2$, then

$$\frac{1}{G_i}\int_{-\infty}^{T_i(t)} g_i(x)\,dx = \int_{-\infty}^{t} 1_{[0,1]}(x)\,dx = t,$$

so the functions u and v in the second proof of Theorem 4.1 are just T_1 and T_2, respectively. In other words, u and v transport a suitable multiple of the characteristic function of the unit interval to g_1 and g_2, respectively.

Barthe saw that this is all that is needed to prove (35) and (36) simultaneously in the special case $n_i = 1$ and $B_i x = x \cdot v_i$, where $x \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$, $i = 1, \ldots, m$. To see this, let $c_i > 0$ satisfy $\sum_i c_i = n$ and let f_i and g_i be nonnegative functions in $L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} f_i(x)\,dx = F_i \text{ and } \int_{\mathbb{R}} g_i(x)\,dx = G_i,$$

for $i = 1, \ldots, m$. Standard approximation arguments show that there is no loss of generality in assuming f_i and g_i are positive and continuous. Define strictly increasing maps T_i as above, so that

$$\frac{1}{F_i}\int_{-\infty}^{t} f_i(x)\,dx = \frac{1}{G_i}\int_{-\infty}^{T_i(t)} g_i(x)\,dx$$

and hence

$$\frac{f_i(x)}{F_i} = \frac{g_i(T_i(x))T_i'(x)}{G_i},$$

for $i = 1, \ldots, m$. For $x \in \mathbb{R}^n$, let

$$V(x) = \sum_{i=1}^{m} c_i T_i(x \cdot v_i) v_i,$$

so that

$$dV(x) = \sum_{i=1}^{m} c_i T_i'(x \cdot v_i)(v_i \otimes v_i)(dx).$$

Finally, note that if $B_i x = x \cdot v_i$ for $x \in \mathbb{R}$, then $B_i^* = x v_i$, so $B_i^* B x = v_i \otimes v_i(x)$, and the constant D in (37) becomes

$$D = \inf\left\{ \frac{\det\left(\sum_{i=1}^{m} c_i a_i v_i \otimes v_i\right)}{\prod_{i=1}^{m} a_i^{c_i}} \,:\, a_i > 0 \right\}.$$

In the following, we can assume that $D \neq 0$. Using the expression for D with $a_i = T_i'(x \cdot v_i)$, $i = 1, \ldots, m$ to provide a lower bound for the Jacobian of the injective map V, we obtain

$$\begin{aligned}
D\left(\prod_{i=1}^{m}\left(\frac{G_i}{F_i}\right)^{c_i}\right)\int_{\mathbb{R}^n}\prod_{i=1}^{m} f_i(x \cdot v_i)^{c_i}\,dx &= D\int_{\mathbb{R}^n}\prod_{i=1}^{m}\left(g_i(T_i(x \cdot v_i))T_i'(x \cdot v_i)\right)^{c_i}\,dx \\
&\leq \int_{\mathbb{R}^n}\prod_{i=1}^{m} g_i(T_i(x \cdot v_i))^{c_i} \det\left(\sum_{i=1}^{m} c_i T_i'(x \cdot v_i)(v_i \otimes v_i)\right)dx \\
&\leq \overline{\int}_{\mathbb{R}^n} \sup\left\{\prod_{i=1}^{m} g_i(z_i)^{c_i} \,:\, V = \sum_i c_i z_i v_i,\, z_i \in \mathbb{R}\right\} dV \\
&\leq \overline{\int}_{\mathbb{R}^n} \sup\left\{\prod_{i=1}^{m} g_i(z_i)^{c_i} \,:\, x = \sum_i c_i z_i v_i,\, z_i \in \mathbb{R}\right\} dx.
\end{aligned}$$

To see how centered Gaussians play a role in the equality conditions, note that if $f_i(x) = \exp(-a_i x^2)$, then since $\sum_i c_i = n$,

$$\begin{aligned}
\prod_{i=1}^{m}\left(\int_{\mathbb{R}} f_i(x)\,dx\right)^{c_i} &= \prod_{i=1}^{m}\left(\int_{\mathbb{R}} e^{-a_i x^2}\,dx\right) \\
&= \prod_{i=1}^{m} a_i^{-c_i/2}\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^{c_i} \\
&= \prod_{i=1}^{m}\left(\frac{\pi}{a_i}\right)^{c_i/2} = \left(\frac{\pi^n}{\prod_{i=1}^{m} a_i^{c_i}}\right)^{1/2},
\end{aligned}$$

while

$$\begin{aligned}\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(x\cdot v_i)^{c_i}\,dx &= \int_{\mathbb{R}^n}\prod_{i=1}^m\left(e^{-a_i(x\cdot v_i)^2}\right)^{c_i}\,dx\\ &= \int_{\mathbb{R}^n} e^{-\left(\sum_{i=1}^m c_i a_i (x\cdot v_i)v_i\right)\cdot x}\,dx\\ &= \left(\frac{\pi^n}{\det\left(\sum_{i=1}^m c_i a_i v_i\otimes v_i\right)}\right)^{1/2}.\end{aligned}$$

(The last equality follows from

$$\int_{\mathbb{R}^n} e^{-Ax\cdot x}\,dx = \left(\frac{\pi^n}{\det A}\right)^{1/2},$$

where A is a positive definite symmetric $n\times n$ matrix.)

To summarize, we have shown that in the special case under consideration, the left-hand side of (36) is greater than or equal to the left-hand side of (35), and that equality holds in (35) for centered Gaussians. This is already enough to prove (36). One more computation is needed to prove (35), but we shall omit it, since it needs some (quite basic) tools of geometry, and refer the reader to [14].

If one wants to apply the same sort of argument in the general situation of Theorem 15.1, one needs an answer to the following question: If μ and ν are measures on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure and with $\mu(\mathbb{R}^n)=\nu(\mathbb{R}^n)$, can we find a T with some suitable monotonicity property that transports μ to ν? It turns out that an ideal answer has recently been found, called the *Brenier map*: Providing μ vanishes on Borel sets of $\mathbb{R}^n$ with Hausdorff dimension $n-1$, there is a convex map $\psi:\mathbb{R}^n\to\mathbb{R}$ such that if $T=\nabla\psi$, then T transports μ to ν. See [17] for more details and references. It is appropriate to highlight the contribution of McCann, whose 1994 PhD thesis [113] shows the relevance of measure-preserving convex gradients to geometric inequalities and helped attract the attention of the convex geometry community to Brenier's result. In [113] and [114], the Brenier map is exploited as a localization technique to derive new global convexity inequalities which imply the Brunn-Minkowski and Prékopa-Leindler inequalities as special cases.

Barthe [15, Section 2.4] also discovered a generalization of Young's inequality in $\mathbb{R}^n$ that contains the geometric Brascamp-Lieb and geometric Barthe inequalities as limiting cases.

18. The entropy power inequality and physics

Suppose that X is a discrete random variable taking possible values $x_1,\dots,x_m$ with probabilities $p_1,\dots,p_m$, respectively, where $\sum_i p_i=1$. Shannon [136] introduced a measure of the average uncertainty removed by revealing the value of X. This quantity,

$$H_m(p_1,\dots,p_m)=-\sum_{i=1}^m p_i\log p_i,$$

is called the *entropy* of X. It can also be regarded as a measure of the missing information; indeed, the function H_m is concave and achieves its maximum when $p_1=\cdots=p_m=1/m$, that is, when all outcomes are equally likely. The words "uncertainty" and "information" already suggest a connection with physics, and a derivation of the function H_m from a few natural assumptions can be found in textbooks on statistical mechanics; see, for example, [6, Chapter 3].

If X is a random vector in $\mathbb{R}^n$ with probability density f, the *entropy* $h_1(X)$ of X is defined analogously:

$$h_1(X) = h_1(f) = -\int_{\mathbb{R}^n} f(x) \log f(x)\, dx.$$

The notation we use is convenient when $h_1(X)$ is regarded as a limit as $p \to 1$ of the pth *Rényi entropy* $h_p(X)$ of X, defined for $p > 1$ by

$$h_p(X) = h_p(f) = \frac{p}{1-p} \log \|f\|_p.$$

The entropy of X may not be well defined. However, if $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p > 1$, then $h_1(X) = h_1(f)$ is well defined, though its value may be $+\infty$.

The *entropy power* $N(X)$ of X is

$$N(X) = \frac{1}{2\pi e} \exp\left(\frac{2}{n} h_1(X)\right).$$

Theorem 18.1. (Entropy power inequality.) *Let X and Y be independent random vectors in $\mathbb{R}^n$ with probability densities in $L^p(\mathbb{R}^n)$ for some $p > 1$. Then*

$$N(X+Y) \geq N(X) + N(Y). \tag{48}$$

The entropy power inequality was proved by Shannon [136, Theorem 15 and Appendix 6] and applied by him to obtain a lower bound [136, Theorem 18] for the capacity of a channel. (Via the web site at `http://www.math.washington.edu/~hillman/Entropy/infcode.html` this paper can be downloaded.) Shannon's proof shows that equality holds in (48) if X and Y are multivariate normal with proportional covariances. In fact equality holds only for such X and Y, as Stam's different proof [138] (simplified in [24] and [48]) of (48) shows.

The most accessible direct proof of (48) seems to be that of Blachman [24]. We present a derivation from Young's inequality and the following lemma, due to Lieb [88].

Lemma 18.2. *Let f and g be nonnegative functions in $L^s(\mathbb{R}^n)$ for some $s > 1$, such that*

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} g(x)\, dx = 1.$$

Then for $0 < \lambda < 1$,

$$h_1(f * g) - (1-\lambda) h_1(f) - \lambda h_1(g) \geq -\frac{n}{2}\left((1-\lambda)\log(1-\lambda) + \lambda \log \lambda\right). \tag{49}$$

Proof. For $r \geq 1$, let

$$p = p(r) = \frac{r}{(1-\lambda) + \lambda r} \text{ and } q = q(r) = \frac{r}{\lambda + (1-\lambda) r}. \tag{50}$$

Then $p, q \geq 1$,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

and $p(1) = q(1) = 1$. If $r < s$ is close to 1, then $p, q < s$, and since $f, g \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$, we have $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Let

$$F(r) = \frac{\|f * g\|_r}{\|f\|_p \|g\|_q} \text{ and } G(r) = C^n,$$

where C is as Theorem 14.1. By Young's inequality (31), $f*g \in L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ and $F(r) \le G(r)$ for r close to 1. As we noted after Theorem 14.1, the equation $F(1) = G(1+)$ holds. Therefore

$$\frac{F(r) - F(1)}{r - 1} \le \frac{G(r) - G(1+)}{r - 1},$$

for r close to 1, which implies that $F'(1+) \le G'(1+)$. We can assume that $h_1(f * g) < \infty$ and therefore that $h_1(f) < \infty$ and $h_1(g) < \infty$. Now if $\phi \in L^r(\mathbb{R}^n)$, $\|\phi\|_1 = 1$, and $h_1(\phi) < \infty$, then

$$\begin{aligned} \frac{d}{dr}\|\phi\|_r &= \frac{1}{r}\|\phi\|^{1-r} \frac{d}{dr} \int_{\mathbb{R}^n} \phi(x)^r \, dx \\ &= \frac{1}{r}\|\phi\|^{1-r} \int_{\mathbb{R}^n} \phi(x)^r \log \phi(x) \, dx \\ &\to \int_{\mathbb{R}^n} \phi(x) \log \phi(x) \, dx = -h_1(\phi) \end{aligned}$$

as $r \to 1$. Using this and (50), we see that

$$F'(1+) = -h_1(f * g) + (1 - \lambda) h_1(f) + \lambda h_1(g).$$

A calculation, helped by the fact that $p' = r'/(1 - \lambda)$ and $q' = r'/\lambda$, where p', q', r' denote as usual the Hölder conjugates of p, q, r, respectively, shows that

$$G'(1+) = \frac{n}{2}\left((1 - \lambda)\log(1 - \lambda) + \lambda \log \lambda\right).$$

Finally, (49) follows from the inequality $F'(1+) \le G'(1+)$. □

Corollary 18.3. *Young's inequality (31) implies the entropy power inequality (48).*

Proof. In (49), put

$$\lambda = \frac{N(Y)}{N(X) + N(Y)}.$$

Simplification of the resulting inequality leads directly to (48). □

Presumably Lieb, via his papers [34] and [88], first saw the connection between the entropy power inequality (48) and the Brunn-Minkowski inequality (15), the former being a limiting case of Young's inequality (31) as $r \to 1$ and the latter a limiting case of the reverse Young inequality (32) as $r \to 0$. Later, Costa and Cover [43] specifically drew attention to the analogy between the two inequalities, apparently unaware of the work of Brascamp and Lieb. Dembo, Cover, and Thomas [48] explore further connections with other inequalities. These include some involving Fisher information and various uncertainty inequalities.

Fisher information was employed by Stam [138] in his proof of (48). Named after the statistician R. A. Fisher, Fisher information is claimed in a recent book [64] by Frieden to be at the heart of a unifying principle for all of physics! If X is a random variable with probability density f on $\mathbb{R}$, the *Fisher information* $I(X)$ of X is

$$I(X) = I(f) = -\int_{\mathbb{R}} f(x)(\log f(x))''\,dx = \int_{\mathbb{R}} \frac{f'(x)^2}{f(x)}\,dx,$$

assuming these integrals exist. The multivariable form of I is a matrix, the natural extension of this definition. The quantity I is another measure of the "sharpness" of f or the missing information in X; see [64, Section 1.3] for a comparison of I and h_1. Stam [138] (see also [48]) showed that I can be used to obtain the Weyl-Heisenberg uncertainty inequality, and this inspired Frieden's work. Frieden's idea is that for any physical system, I represents how much information can possibly be obtained by measurements, while another quantity, J, is the amount of information bound up in the system. Then $I-J$ leads to a Lagrangian, and the corresponding law of physics arises from its minimization, the second derivative usually present in such a law arising from the first derivative present in I.

Needless to say, Frieden's claim has stirred some controversy. Some opinions can be found on the web site at `http://members.home.net/stephenk1/Outlaw/fisherinfo.html` and in the *Mathematical Reviews* review.

The many related inequalities involving entropy and Fisher information are also connected to other consequences of Young's inequality, such as Nelson's hypercontractive inequality and various logarithmic Sobolev inequalities; see [48] and Section 19.14. The papers [74], [106], and [108] provide still more connections between information theory and convex geometry.

19. A survey

In the subsections below we attempt an overview of the various known extensions and analogs of the Brunn-Minkowski inequality not covered above. Without being comprehensive, it should alert the reader to the main developments.

19.1. **The Aleksandrov-Fenchel inequality.**

Theorem 19.1. (Aleksandrov-Fenchel inequality.) *Let $K_1,\dots,K_n$ be compact convex sets in $\mathbb{R}^n$ and let $1\le i\le n$. Then*

$$V(K_1,K_2,\dots,K_n)^i \ge \prod_{j=1}^{i} V(K_j,i;K_{i+1},\dots,K_n)\,. \tag{51}$$

See [37, p. 143] and [134, (6.8.7)]. The quantities $V(K_1,K_2,\dots,K_n)$ and $V(K_j,i;K_{i+1},\dots,K_n)$ (where the notation means that K_j appears i times) are mixed volumes, like the quantity $V_1(K,L)$ we met in Section 7. In fact, if we put $i=n$ in (51) and then let $K_1=L$ and $K_2=\cdots=K_n=K$, we retrieve Minkowski's first inequality (20) for compact convex sets. Therefore the Aleksandrov-Fenchel inequality implies the Brunn-Minkowski inequality for compact convex sets. In fact, there is a more general version of the latter that is equivalent to (51):

Theorem 19.2. (Generalized Brunn-Minkowski inequality for compact convex sets.) *Let $K_1,\dots,K_n$ be compact convex sets in $\mathbb{R}^n$ and let $1\le i\le n$. For $0\le\lambda\le1$, let*

$$f(\lambda) = V((1-\lambda)K_0+\lambda K_1,i;K_{i+1},\dots,K_n)^{1/i}\,.$$

Then f is a concave function on $[0,1]$.

See [37, p. 146] and [134, Theorem 6.4.3]. The Brunn-Minkowski inequality for compact convex sets is the case $i = n$ of Theorem 19.2. Readers familiar with the basic properties of mixed volumes can derive (51) by setting $i = 2$ in Theorem 19.2 and expanding the resulting inequality to extract the constants $(1 - \lambda)$ and λ. Inequality (51) with $i = 2$ results, and the general case follows by induction on i.

For compact convex sets, (51) is essentially the most powerful extension of the Brunn-Minkowski inequality known. Proofs of Theorems 19.1 and 19.2, discovered by A. D. Aleksandrov and by W. Fenchel and B. Jessen independently around 1937, can be found in [134, Theorems 6.3.1 and 6.4.3]. Equality conditions are not fully settled even today. An analog of the Aleksandrov-Fenchel inequality for mixed discriminants (see [134, Theorem 6.8.1]) was used by G. P. Egorychev in 1981 to solve the van der Waerden conjecture concerning the permanent of a doubly stochastic matrix. See [134, Chapter 6] for a wealth of information and references.

Khovanskii, who with Teissier independently discovered that the Aleksandrov-Fenchel inequality can be deduced from the Hodge index theorem, wrote a readable account of this surprising development in [37, Section 27]. The connection originates in the fact (due to D. M. Bernstein) that the number of complex roots of a generic system of n polynomial equations in n variables equals $n!$ times the mixed volume of the corresponding Newton polytopes, $P_1, P_2, \ldots P_n$, say. (The Newton polytope is the smallest convex polytope in $\mathbb{R}^n$ containing each point $(m_1, \ldots, m_n)$ for which $cz_1^{m_1} \cdots z_n^{m_n}$ is a term of the polynomial.) The $(n - 2)$ of these n polynomial equations corresponding to $P_3, \ldots, P_n$ define an algebraic surface in $\mathbb{C}^n$ on which the remaining polynomial equations describe two complex curves. The number of intersection points of these two curves is the number of roots of the system of n equations. Roughly speaking, the Hodge index theorem is an inequality involving the number of intersections of two complex curves Γ_1, Γ_2 in a compact complex algebraic surface and those of each curve with a slightly deformed copy of itself:

$$\langle \Gamma_1, \Gamma_2 \rangle^2 \geq \langle \Gamma_1, \Gamma_1 \rangle \langle \Gamma_2, \Gamma_2 \rangle.$$

Using the above observations, this can be translated into

$$V(P_1, P_2, P_3, \ldots, P_n)^2 \geq V(P_1, P_1, P_3, \ldots, P_n) V(P_2, P_2, P_3, \ldots, P_n).$$

The case $i = 2$ of (19.1) (and hence, by induction, (19.1) itself) can be shown to follow by approximation by polytopes with rational coordinates. See [37, Section 27] for many more details and also [72] and [123] for more recent advances in this direction.

Alesker, Dar, and Milman [1] are able to use the Brenier map (see the end of Section 17) to prove some of the inequalities that follow from the Aleksandrov-Fenchel inequality, but the method does not seem to yield a new proof of (51) itself.

In contrast to the Brunn-Minkowski inequality, the Aleksandrov-Fenchel inequality and some of its weaker forms, and indeed mixed volumes themselves, have only partially successful extensions to nonconvex sets. See [37, pp. 177–181], [134, p. 343], and [145].

19.2. **Minkowski-concave functions.** A real-valued function ϕ defined on a class of sets in $\mathbb{R}^n$ closed under Minkowski addition and dilatation is called *Minkowski concave* if

$$\phi((1 - \lambda)X + \lambda Y) \geq (1 - \lambda)\phi(X) + \lambda\phi(Y), \tag{52}$$

for $0 < \lambda < 1$ and sets X, Y in the class. For example, the Brunn-Minkowski inequality implies that $V_n^{1/n}$ is Minkowski concave on the class of convex bodies. When Hadwiger published his extraordinary book [75] in 1957, many other Minkowski-concave functions had already been found, and several more have been discovered since. We shall present some of these; all the functions have the required degree of positive homogeneity to allow the coefficients $(1-\lambda)$ and λ to be deleted in (52). Other examples can be found in [75, Section 6.4] and in Lutwak's papers [96] and [102].

Knothe [83] gave a proof of the Brunn-Minkowski inequality for convex bodies, sketched in [134, pp. 312–314], and the following generalization. For each convex body K in $\mathbb{R}^n$, let $F(K,x)$, $x \in K$, be a nonnegative real-valued function continuous in K and x. Suppose also that for some $m > 0$,

$$F(\lambda K + a, \lambda x + a) = \lambda^m F(K,x)$$

for all $\lambda > 0$ and $a \in \mathbb{R}^n$, and that

$$\log F\left((1-\lambda)K + \lambda L, (1-\lambda)x + \lambda y\right) \geq (1-\lambda)\log F(K,x) + \lambda \log F(L,y)$$

whenever $x \in K$, $y \in L$, and $0 \leq \lambda \leq 1$. For each convex body K in $\mathbb{R}^n$, define

$$G(K) = \int_K F(K,x)\,dx.$$

Then

$$G(K+L)^{1/(n+m)} \geq G(K)^{1/(n+m)} + G(L)^{1/(n+m)}, \tag{53}$$

for all convex bodies K and L in $\mathbb{R}^n$ and $0 < \lambda < 1$. This is a consequence of the Prékopa-Leindler inequality. Indeed, taking $f = F((1-\lambda)K + \lambda L, \cdot)$, $g = F(K, \cdot)$, and $h = F(L, \cdot)$, Theorem 4.2 implies that G is log concave. The method of Section 5 can then be used to derive the $1/(n+m)$-concavity (53) of G from its log concavity. The Brunn-Minkowski inequality for convex bodies is obtained by taking $F(K,x) = 1$ for $x \in K$. Dinghas [50] found further results of this type.

Let $0 \leq i \leq n$. The mixed volume $V(K, n-i; B, i)$ is denoted by $W_i(K)$, and called the ith *quermassintegral* of a compact convex set K in $\mathbb{R}^n$. Then $W_0(K) = V_n(K)$. It can be shown (see [134, (5.3.27), p. 295]) that if K is a convex body and $1 \leq i \leq n-1$, then

$$W_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \int_{G(n,n-i)} V(K|S)\,dS, \tag{54}$$

where dS denotes integration with respect to the usual rotation-invariant probability measure on the Grassmannian $G(n, n-i)$ of $(n-i)$-dimensional subspaces of $\mathbb{R}^n$. Thus the quermassintegrals are averages of volumes of projections on subspaces.

Letting $K_{i+1} = \cdots = K_n = B$ in Theorem 19.2 yields:

Theorem 19.3. (Brunn-Minkowski inequality for quermassintegrals.) *Let K and L be convex bodies in $\mathbb{R}^n$ and let $0 \leq i \leq n-1$. Then*

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{55}$$

with equality for $0 < i < n-1$ if and only if K and L are homothetic.

See [134, (6.8.10), p. 385]. The special case $i = 0$ is the usual Brunn-Minkowski inequality for convex bodies. The quermassintegral $W_1(K)$ equals the surface area $S(K)$, up to a constant, so the case $i = 1$ of (55) is a Brunn-Minkowski-type inequality for surface area. When $i = n-1$, (55) becomes an identity. The equality conditions for $0 < i < n-1$ follow from those known for the corresponding special case of Theorem 19.2.

Let K be a convex body in $\mathbb{R}^n$, define $\hat{W}_0(K) = V(K)$ and for $1 \le i \le n-1$, define

$$\hat{W}_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\int_{G(n,n-i)} V(K|S)^{-1} \, dS \right)^{-1},$$

the ith *harmonic quermassintegral* of K. Similarly, define $\Phi_0(K) = V(K)$ and for $1 \le i \le n-1$, define

$$\Phi_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \left(\int_{G(n,n-i)} V(K|S)^{-n} \, dS \right)^{-1/n},$$

the ith *affine quermassintegral* of K. Note the similarity to (54); the ordinary mean has been replaced by the -1- and $-n$-means, respectively. As its name suggests, $\Phi_i(K)$ is invariant under volume-preserving affine transformations. Hadwiger [75, p. 268] proved the following inequality.

Theorem 19.4. (Hadwiger's inequality for harmonic quermassintegrals.) *If K and L are convex bodies in $\mathbb{R}^n$ and $0 \le i \le n-1$, then*

$$\hat{W}_i(K+L)^{1/(n-i)} \ge \hat{W}_i(K)^{1/(n-i)} + \hat{W}_i(L)^{1/(n-i)}.$$

Lutwak [97] showed that the same inequality holds for affine quermassintegrals.

Theorem 19.5. (Lutwak's inequality for affine quermassintegrals.) *If K and L are convex bodies in $\mathbb{R}^n$ and $0 \le i \le n-1$, then*

$$\Phi_i(K+L)^{1/(n-i)} \ge \Phi_i(K)^{1/(n-i)} + \Phi_i(L)^{1/(n-i)}. \tag{56}$$

Let K be a convex body in $\mathbb{R}^n$, $n \ge 3$. The *capacity* $\mathrm{Cap}\,(K)$ of K is defined by

$$\mathrm{Cap}\,(K) = \inf \left\{ \int_{\mathbb{R}^n} \|\nabla f\|^2 \, dx : \; f \in C_c^\infty(\mathbb{R}^n), \; f \ge 1_K \right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Here we are following Evans and Gariepy [57, p. 147], where $\mathrm{Cap}\,(K) = \mathrm{Cap}_{\,n-2}(K)$ in their notation. Several definitions are possible; see [79] and [111, pp. 110–116]. The notion of capacity has its roots in electrostatics and is fundamental in potential theory. Note that capacity is an outer measure but is *not* a Borel measure, though it enjoys some convenient properties listed in [57, p. 151].

Borell [29] proved the following theorem.

Theorem 19.6. (Borell's inequality for capacity.) *If K and L are convex bodies in $\mathbb{R}^n$, $n \ge 3$, then*

$$\mathrm{Cap}\,(K+L)^{1/(n-2)} \ge \mathrm{Cap}\,(K)^{1/(n-2)} + \mathrm{Cap}\,(L)^{1/(n-2)}. \tag{57}$$

Caffarelli, Jerison, and Lieb [39] showed that equality holds if and only if K and L are homothetic. Jerison [79] employed the inequality and its equality conditions in solving the corresponding Minkowski problem (see Section 7).

19.3. **Blaschke addition.** If K and L are convex bodies in $\mathbb{R}^n$, then there is a convex body $K \dotplus L$ such that

$$S(K \dotplus L, \cdot) = S(K, \cdot) + S(L, \cdot),$$

where $S(K, \cdot)$ denotes the surface area measure of K. This is a consequence of Minkowski's existence theorem; see [67, Theorem A.3.2] or [134, Section 7.1]. The operation $\dotplus$ is called *Blaschke addition*.

Theorem 19.7. (Kneser-Süss inequality.) *If K and L are convex bodies in $\mathbb{R}^n$, then*

$$V(K \dotplus L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \tag{58}$$

with equality if and only if K and L are homothetic.

See [134, Theorem 7.1.3] for a proof.

Using Blaschke addition, a convex body called a mixed body can be defined from $(n-1)$ other convex bodies in $\mathbb{R}^n$. Lutwak [98, Theorem 4.2] exploits this idea, due to Blaschke and Firey, to produce another strengthening of the Brunn-Minkowski inequality for convex bodies.

19.4. **The L^p-Brunn-Minkowski theory.** For convex bodies K and L in $\mathbb{R}^n$, Minkowski addition can be defined by

$$h_{K+L}(u) = h_K(u) + h_L(u),$$

for $u \in S^{n-1}$, where h_K denotes the support function of K. If $p \geq 1$ and K and L contain the origin in their interiors, a convex body $K +_p L$ can be defined by

$$h_{K+_pL}(u)^p = h_K(u)^p + h_L(u)^p,$$

for $u \in S^{n-1}$. The operation $+_p$ is called *p-Minkowski addition*. Firey [60] proved the following inequality. (Both the definition of p-Minkowski addition and the case $i = 0$ of Firey's inequality are extended to nonconvex sets by Lutwak, Yang, and Zhang [105].)

Theorem 19.8. (Firey's inequality.) *If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, $0 \leq i \leq n-1$ and $p \geq 1$, then*

$$W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{59}$$

with equality when $p > 1$ if and only if K and L are equivalent by dilatation.

The Brunn-Minkowski inequality for quermassintegrals (55) is the case $p = 1$. Note that translation invariance is lost for $p > 1$.

Firey's ideas were transformed into a remarkable extension of the Brunn-Minkowski theory by Lutwak [101], [104], who also calls it the *Brunn-Minkowski-Firey theory*. Lutwak found the appropriate p-analog $S_p(K, \cdot)$, $p \geq 1$, of the surface area measure of a convex body K in $\mathbb{R}^n$ containing the origin in its interior. In [101], Lutwak generalized Firey's inequality (59). He also generalized Minkowski's existence theorem, deduced the existence of a convex body $K \dotplus_p L$ for which

$$S_p(K \dotplus_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot)$$

(when K and L are origin-symmetric convex bodies), and proved the following result.

Theorem 19.9. (Lutwak's p-surface area measure inequality.) *If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ and $n \neq p \geq 1$, then*

$$V(K \dotplus_p L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n},$$

with equality when $p > 1$ if and only if K and L are equivalent by dilatation.

Note that the Kneser-Süss inequality (58) corresponds to $p = 1$.

Lutwak, Yang, and Zhang [107] study the L^p version of the Minkowski problem (see Section 7). A version corresponding to $p = 0$ is treated by Stancu [139].

19.5. **Random and integral versions.** Let $\mathcal{X}$ be a random set in $\mathbb{R}^n$, that is, a Borel measurable map from a probability space Ω to the space of nonempty compact sets in $\mathbb{R}^n$ with the Hausdorff metric. A random vector $X : \Omega \to \mathbb{R}^n$ is called a *selection* of $\mathcal{X}$ if $\mathrm{Prob}\,(X \in \mathcal{X}) = 1$. If C is a nonempty compact set in $\mathbb{R}^n$, let $\|C\| = \max\{\|x\| : x \in C\}$. Then the expectation $E\mathcal{X}$ of X is defined by

$$E\mathcal{X} = \{EX : X \text{ is a selection of } \mathcal{X} \text{ and } E\|X\| < \infty\}.$$

It turns out that if $E\|\mathcal{X}\| < \infty$, then $E\mathcal{X}$ is a nonempty compact set.

Theorem 19.10. (Vitale's random Brunn-Minkowski inequality.) *Let $\mathcal{X}$ be a random set in $\mathbb{R}^n$ with $E\|\mathcal{X}\| < \infty$. Then*

$$V_n(E\mathcal{X})^{1/n} \geq EV_n(\mathcal{X})^{1/n}. \tag{60}$$

See [148] (and [149] for a stronger version). By taking $\mathcal{X}$ to be a random set that realizes values (nonempty compact sets) K and L with probabilities $(1 - \lambda)$ and λ, respectively, we see that Theorem 19.10 generalizes the Brunn-Minkowski inequality for compact sets.

A version of (60) for intrinsic volumes (weighted quermassintegrals) of random convex bodies, and applications to stationary random hyperplane processes, are given by Mecke and Schwella [117].

Earlier integral forms of the Brunn-Minkowski inequality, using a Riemann approach to pass from a Minkowski sum to a "Minkowski integral," were formulated by A. Dinghas; see [37, p. 76].

19.6. **Other strong forms of the Brunn-Minkowski inequality for convex sets.** McMullen [116] defines a natural generalization of Minkowski addition of convex sets that he calls fibre addition, and proves a corresponding Brunn-Minkowski inequality.

Several strong forms of the Brunn-Minkowski inequality hold in special circumstances, for example, the stability estimates due to V. Diskant, H. Groemer, and R. Schneider referred to in [71, Section 3] and [134, p. 314], and an inequality of Ruzsa [131].

Dar [46] conjectures that if K and L are convex bodies in $\mathbb{R}^n$ and $m = \max_{x\in\mathbb{R}^n} V(K\cap(L+x))$, then

$$V(K + L)^{1/n} \geq m^{1/n} + \left(\frac{V(K)V(L)}{m}\right)^{1/n}. \tag{61}$$

He shows that (61) implies the Brunn-Minkowski inequality for convex bodies and proves that it holds in some special cases.

19.7. **Related affine inequalities.** A wide variety of fascinating inequalities lie (for the present) one step removed from the Brunn-Minkowski inequality. The survey paper [124] of Osserman indicates connections between the isoperimetric inequality and inequalities of Bonnesen, Poincaré, and Wirtinger, and since then many other inequalities have been found that lie in a complicated web around the Brunn-Minkowski inequality.

Some of these related inequalities are *affine inequalities* in the sense that they are unchanged under a volume-preserving linear transformation. The Brunn-Minkowski and Prékopa-Leindler inequalities are clearly affine inequalities. Young's inequality and its reverse are affine inequalities, since if $\phi \in SL(n)$, we have

$$\phi(f * g) = (\phi f) * (\phi g) \text{ and } \|\phi f\|_p = \|f\|_p.$$

The Brascamp-Lieb and Barthe inequalities are also affine inequalities.

The sharp Hardy-Littlewood-Sobolev inequality (39) is not affine invariant, but it is invariant under conformal transformations; see [91, Theorem 4.5]. The isoperimetric inequality is also not an affine inequality (if it were, the equality for balls would imply that equality also held for ellipsoids), and neither is the Sobolev inequality (24).

There is a remarkable affine inequality that is much stronger than the isoperimetric inequality for convex bodies. The *Petty projection inequality* states that

$$V(K)^{n-1}V(\Pi^* K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n, \tag{62}$$

where K is a convex body in $\mathbb{R}^n$, and $\Pi^* K$ denotes the polar body of the projection body ΠK of K. (The support function of ΠK at $u \in S^{n-1}$ equals $V(K|u^{\perp})$.) Equality holds if and only if K is an ellipsoid. See [67, Chapter 9] for background information, a proof, several other related inequalities, and a reverse form due to Zhang. Zhang [152] has also recently found an astounding affine Sobolev inequality, a common generalization of the Sobolev inequality (24) and the Petty projection inequality (62): If $f \in C^1(\mathbb{R}^n)$ has compact support, then

$$\left(\int_{S^{n-1}} \|D_u f\|_1^{-n}\, du\right)^{-1/n} \geq \frac{2\kappa_{n-1}}{n^{1/n}\kappa_n}\|f\|_{n/(n-1)}, \tag{63}$$

where $D_u f$ is the directional derivative of f in the direction u.

This is only a taste of a banquet of known affine isoperimetric inequalities. Lutwak [103] wrote an excellent survey. For still more recent progress, the reader can do no better than consult the work of Lutwak, Yang, and Zhang, for example, [109] and [110].

19.8. **A restricted Brunn-Minkowski inequality.** Let X and Y be measurable sets in $\mathbb{R}^n$, and let E be a measurable subset of $X \times Y$. Define the *restricted Minkowski sum* of X and Y by

$$X +_E Y = \{x + y : \ (x, y) \in E\}.$$

Theorem 19.11. (Restricted Brunn-Minkowski inequality.) *There is a $c > 0$ such that if X and Y are nonempty measurable subsets of $\mathbb{R}^n$, $0 < t < 1$,*

$$t \leq \left(\frac{V_n(X)}{V_n(Y)}\right)^{1/n} \leq \frac{1}{t}, \text{ and } \frac{V_n(E)}{V_n(X)V_n(Y)} \geq 1 - c\min\{t\sqrt{n}, 1\},$$

then

$$V_n(X +_E Y)^{2/n} \geq V_n(X)^{2/n} + V_n(Y)^{2/n}.$$

Szarek and Voiculescu [142] proved Theorem 19.11 in the course of establishing an analog of the entropy power inequality in Voiculescu's free probability theory. (Voiculescu has also found analogs of Fisher information within this noncommutative probability theory with applications to physics.) Barthe [19] also gives a proof via restricted versions of Young's inequality and the Prékopa-Leindler inequality.

19.9. **Milman's reverse Brunn-Minkowski inequality.** At first such an inequality seems impossible, since if K and L are convex bodies in $\mathbb{R}^n$ of volume 1, the volume of $K+L$ can be arbitrarily large. As with the reverse isoperimetric inequality (45), however, linear transformations come to the rescue.

Theorem 19.12. (Milman's reverse Brunn-Minkowski inequality.) *There is a constant c independent of n such that if K and L are centrally symmetric convex bodies in $\mathbb{R}^n$, there are volume-preserving linear transformations ϕ and ψ for which*

$$V(\phi K+\psi L)^{1/n} \le c\left(V(\phi K)^{1/n}+V(\psi L)^{1/n}\right). \tag{64}$$

First proved by V. Milman in 1986, this result is important in the local theory of Banach spaces. See [92, Section 4.3] and [127, Chapter 7].

19.10. **Discrete versions.** The *Cauchy-Davenport theorem*, proved by Cauchy in 1813 and rediscovered by Davenport in 1935, states that if p is prime and X and Y are nonempty finite subsets of $\mathbb{Z}/p\mathbb{Z}$, then

$$|X+Y| \ge \min\{p, |X|+|Y|-1\}.$$

Here $|X|$ is the cardinality of X. Many generalizations of this result, including Kneser's extension to Abelian groups, are surveyed in [122]. The lower bound for a vector sum is in the spirit of the Brunn-Minkowski inequality. We now describe a closer analog.

Let Y be a finite subset of $\mathbb{Z}^n$ with $|Y| \ge n+1$. For $x=(x_1,\dots,x_n)\in\mathbb{Z}^n$, let

$$w_Y(x)=\frac{x_1}{|Y|-n}+\sum_{i=2}^n x_i.$$

Define the Y*-order* on $\mathbb{Z}^n$ by setting $x <_Y y$ if either $w_Y(x)<w_Y(y)$ or $w_Y(x)=w_Y(y)$ and for some j we have $x_j>y_j$ and $x_i=y_i$ for all $i<j$. For $m\in\mathbb{N}$, let D^Y_m be the union of the first m points in $\mathbb{Z}^n_+$ (the points in $\mathbb{Z}^n$ with nonnegative coordinates) in the Y-order. The set D^Y_m is called a Y*-initial segment*. The points of $D^Y_{|Y|}$ are

$$o <_Y e_1 <_Y 2e_1 <_Y \cdots <_Y (|Y|-n)e_1 <_Y e_2 <_Y \cdots <_Y e_n,$$

where $e_1,\dots,e_n$ is the standard orthonormal basis for $\mathbb{R}^n$. Note that the convex hull of $D^Y_{|Y|}$ is a simplex. Roughly speaking, Y-initial segments are as close as possible to being the set of points in $\mathbb{Z}^n_+$ that are contained in a dilatate of this simplex.

Theorem 19.13. (Brunn-Minkowski inequality for the integer lattice.) *Let X and Y be finite subsets of $\mathbb{Z}^n$ with* $\dim Y=n$. *Then*

$$|X+Y| \ge \left|D^Y_{|X|}+D^Y_{|Y|}\right|. \tag{65}$$

See [68], and also [26] for a similar result in finite subgrids of $\mathbb{Z}^n$. That (65) is indeed a Brunn-Minkowski-type inequality is clear by comparing

$$V(K+L) \geq V(r_K B + r_L B),$$

the consequence of (17) given above. Indeed, (65) is proved by means of a discrete version, called compression, of an anti-symmetrization process related to Steiner symmetrization.

19.11. **The dual Brunn-Minkowski theory.** Let M be a body in $\mathbb{R}^n$ containing the origin in its interior and star-shaped with respect to the origin. The *radial function* of M is defined by

$$\rho_M(u) = \max\{c : cu \in M\},$$

for $u \in S^{n-1}$. Call M a *star body* if ρ_M is positive and continuous on S^{n-1}.

Let M and N be star bodies in $\mathbb{R}^n$, let $p \neq 0$, and define a star body $M \widetilde{+}_p N$ by

$$\rho_{M\widetilde{+}_p N}(u)^p = \rho_M(u)^p + \rho_N(u)^p.$$

The operation $\widetilde{+}_p$ is called *p-radial addition*.

Theorem 19.14. (p-dual Brunn-Minkowski inequality.) *If M and N are star bodies in $\mathbb{R}^n$, and $0 < p \leq n$, then*

$$V(M\widetilde{+}_p N)^{p/n} \leq V(M)^{p/n} + V(N)^{p/n}. \tag{66}$$

The reverse inequality holds when $p > n$ or when $p < 0$. Equality holds when $p \neq n$ if and only if M and N are equivalent by dilatation.

The inequality (66) follows from the polar coordinate formula for volume and Minkowski's integral inequality (see [77, Section 6.13]). It was found by Firey [59] for convex bodies and $p \leq -1$. The general inequality forms part of Lutwak's highly successful dual Brunn-Minkowski theory, in which the intersections of star bodies with subspaces replace the projections of convex bodies onto subspaces in the classical theory; see, for example, [67]. The cases $p = 1$ and $p = n-1$ are called the *dual Brunn-Minkowski inequality* and *dual Kneser-Süss inequality*, respectively. A renormalized version of the case $p = n+1$ of (66) was used by Lutwak [100] in his work on centroid bodies (see also [67, Section 9.1]).

There is an inequality equivalent to the dual Brunn-Minkowski inequality called the *dual Minkowski inequality*, the analog of Minkowski's first inequality (20); see [67, p. 373]. This plays a role in the solution of the Busemann-Petty problem (the analog of Shephard's problem mentioned after Theorem 7.1): If the intersection of an origin-symmetric convex body with any given hyperplane containing the origin is always smaller in volume than that of another such body, is its volume also smaller? The answer is no in general in five or more dimensions, but yes in less than five dimensions. See [65], [66], [69], [151], and [153].

Lutwak [95] also discovered that integrals over S^{n-1} of products of radial functions behave like mixed volumes, and called them dual mixed volumes. In the same paper, he showed that a suitable version of Hölder's inequality in S^{n-1} then becomes a dual form of the Aleksandrov-Fenchel inequality (51), in which mixed volumes are replaced by dual mixed volumes (and the inequality is reversed). Special cases of dual mixed volumes analogous to the quermassintegrals are called dual quermassintegrals, and it can be shown that an expression similar to (54) holds for these; instead of averaging volumes of projections, this involves averaging volumes of intersections with subspaces. Dual affine quermassintegrals can also be defined (see [67, p. 332]), but apparently an inequality for these corresponding to (56) is not known.

19.12. **Busemann's theorem.** Let S be an $(n-2)$-dimensional subspace of $\mathbb{R}^n$, let $u \in S^{n-1} \cap S^\perp$, and let S_u denote the closed $(n-1)$-dimensional half-subspace containing u and with S as boundary. Let $u, v \in S^{n-1} \cap S^\perp$, and let X and Y be subsets of S_u and S_v, respectively. If $0 < \lambda < 1$, let $u(\lambda)$ be the unit vector in the direction $(1-\lambda)u + \lambda v$, and let $(1-\lambda)X +_h \lambda Y$ be the set of points in $S_{u(\lambda)}$ lying on a line segment with one endpoint in X and the other in Y. We call the operation $+_h$ *harmonic addition.*

Theorem 19.15. (Busemann-Barthel-Franz inequality.) *In the notation introduced above, let X and Y be compact subsets of S_u and S_v, respectively, of positive V_{n-1}-measure. If $0 < \lambda < 1$, then*

$$\frac{V_{n-1}\left((1-\lambda)X +_h \lambda Y\right)}{\|u(\lambda)\|} \geq M_{-1}(V_{n-1}(X), V_{n-1}(Y), \lambda). \tag{67}$$

Though Theorem 19.15 looks strange, it has the following nice geometrical consequence called *Busemann's theorem.* If K is a convex body in $\mathbb{R}^n$ containing the origin in its interior and S is an $(n-2)$-dimensional subspace, the curve $r = r(\theta)$ in $S^\perp$ such that $r(\theta)$ is the $(n-1)$-dimensional volume of the intersection of K with the half-space S_θ forms the boundary of a convex body in $S^\perp$. Proved in this form by H. Busemann in 1949 and motivated by his theory of area in Finsler spaces, it is also important in geometric tomography (see [67, Theorem 8.1.10]). As stated, Theorem 19.15 and precise equality conditions were proved by W. Barthel and G. Franz in 1961; see [67, Note 8.1] for more details and references.

Milman and Pajor [119, Theorem 3.9] found a proof of Busemann's theorem similar to the second proof of Theorem 4.1 given above. Generalizations along the lines of Theorem 10.2 are possible, such as the following (stated and proved in [15, p. 9]).

Theorem 19.16. *Let $0 < \lambda < 1$, let $p > 0$, and let f, g, and h be nonnegative integrable functions on $[0, \infty)$ satisfying*

$$h\left(M_{-p}(x, y, \lambda)\right) \geq f(x)^{\frac{(1-\lambda)y^p}{(1-\lambda)y^p + \lambda x^p}} g(y)^{\frac{\lambda x^p}{(1-\lambda)y^p + \lambda x^p}}, \tag{68}$$

for all nonnegative $x, y \in \mathbb{R}$. Then

$$\int_0^\infty h(x)\,dx \geq M_{-p}\left(\int_0^\infty f(x)\,dx, \int_0^\infty g(x)\,dx, \lambda\right).$$

The previous inequality is very closely related to one found earlier by Ball [8]. For other associated inequalities, see [70, Theorem 4.1] and [118, Lemma 1].

19.13. **Brunn-Minkowski and Prékopa-Leindler inequalities in other spaces.** Let X be a measurable subset of $\mathbb{R}^n$ and let r_X be the radius of a ball of the same volume as X. If $\varepsilon > 0$, the Brunn-Minkowski inequality (16) implies that

$$V_n(X + \varepsilon B) \geq \left(V_n(X)^{1/n} + \varepsilon V_n(B)^{1/n}\right)^n = \left(V_n(r_X B)^{1/n} + \varepsilon V_n(B)^{1/n}\right)^n = V_n(r_X B + \varepsilon B). \tag{69}$$

For any set A, write

$$A_\varepsilon = A + \varepsilon B = \{x : d(x, A) \leq \varepsilon\}. \tag{70}$$

Then we can rewrite (69) as

$$V_n(X_\varepsilon) \geq V_n((r_X B)_\varepsilon). \tag{71}$$

Notice that (71), by virtue of (70), is now free of the addition and involves only a measure and a metric.

With the appropriate measure and metric replacing V_n and the Euclidean metric, (71) remains true in the sphere S^{n-1} and hyperbolic space, equality holding if and only if X is a ball of radius r_X. (Of course, in these spaces, the ball $B(x,r)$ centered at x and with radius $r > 0$ is the set of all points whose distance from x is at most r. In S^{n-1}, balls are just spherical caps.) Though in $\mathbb{R}^n$ (71) is only a special case of (16), in S^{n-1} and hyperbolic space, (71) is called the Brunn-Minkowski inequality. According to Dudley [52, p. 184], (71) was first proved in S^{n-1} under extra assumptions by P. Lévy in 1922, with weaker assumptions by E. Schmidt in the 1940's, and in full generality by Figiel, Lindenstrauss, and Milman in 1977. In hyperbolic space, (71) is due to E. Schmidt. A proof using symmetrization techniques for both S^{n-1} and hyperbolic space can be found in [37, Section 9].

Perhaps more significant than (71) for recent developments is a surprising result that holds in S^{n-1}, $n \geq 3$, with the chordal metric. It can be shown that if $V_{n-1}(X)/V_{n-1}(B) \geq 1/2$ and $0 < \varepsilon < 1$, then

$$\frac{V_{n-1}(X_\varepsilon)}{V_{n-1}(B)} \geq 1 - \left(\frac{\pi}{8}\right)^{1/2} e^{-(n-2)\varepsilon^2/2}. \tag{72}$$

Results of the form (72) are called *approximate isoperimetric inequalities*, and can be derived from the general Brunn-Minkowski inequality in $\mathbb{R}^n$, as in [4, Theorem 2]. In particular, by taking X to be a hemisphere, we see that for large n, almost all the measure is concentrated near the equator! This result, which again goes back to P. Lévy, is proved in [120, p. 5]. It is an example of the *concentration of measure* phenomenon that Milman applied in his 1971 proof of Dvoretzky's theorem, and that with contributions by Talagrand and others has quickly generated an extensive literature surveyed by Ledoux [85]. An excellent, but more selective, introduction is Ball's elegant and insightful expository article [13, Lecture 8].

Analogous results hold in Gauss space, $\mathbb{R}^n$ with the usual metric but with the standard *Gauss measure* γ_n in $\mathbb{R}^n$ with density

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-\|x\|^2/2}\, dx.$$

Indeed, for bounded Lebesgue measurable sets X and Y in $\mathbb{R}^n$ for which $(1-\lambda)X + \lambda Y$ is Lebesgue measurable, we have the inequality

$$\gamma_n((1-\lambda)X + \lambda Y) \geq \gamma_n(X)^{1-\lambda} \gamma_n(Y)^\lambda \tag{73}$$

corresponding to (14). This follows from the Prékopa-Leindler inequality (because the density function is log concave); see, for example, [33]. It can also be derived directly from the general Brunn-Minkowski inequality in $\mathbb{R}^n$ by means of the "Poincaré limit," a limit of projections of Lebesgue measure in balls of increasing radius; this and an abundance of additional information and references can be found in Ledoux and Talagrand's book [86, Section 1.1]. To describe some of this work briefly, let $\Phi(r) = \gamma_1((-\infty, r))$ for $r \in \mathbb{R}$. Borell [27] and Sudakov and Tsirel'son [141] independently showed that if X is a measurable subset of $\mathbb{R}^n$ and $\gamma_n(X) = \Phi(r_X)$, then $\gamma_n(X_\varepsilon) \geq \Phi(r_X + \varepsilon)$, with equality if X is a half-space. Ehrhard [54], [55] gave a new proof using symmetrization techniques that also yields the following Brunn-Minkowski-type inequality. If K and L are convex bodies in $\mathbb{R}^n$ and $0 < \lambda < 1$, then

$$\Phi^{-1}\left(\gamma_n((1-\lambda)K + \lambda L)\right) \geq (1-\lambda)\Phi^{-1}\left(\gamma_n(K)\right) + \lambda\Phi^{-1}\left(\gamma_n(L)\right). \tag{74}$$

While (74) is stronger than (73) for convex bodies, it is unknown whether it holds for Borel sets; see [84] and [86, Problem 1]. An approximate isoperimetric inequality similar to (72) also holds in Gauss space; Maurey [112] (see also see [13, Theorem 8.1]) found a simple proof employing the Prékopa-Leindler inequality. As in S^{n-1}, there is a concentration of measure in Gauss space, this time in spherical shells of thickness approximately 1 and radius approximately $\sqrt{n}$. Closely related work on logarithmic Sobolev inequalities is outlined in the next section.

Bahn and Ehrlich [5] find an inequality that can be interpreted as a reversed form of the Brunn-Minkowski inequality in Minkowski spacetime, that is, $\mathbb{R}^{n+1}$ with a scalar product of index 1.

Cordero-Erausquin [41] utilizes results of R. McCann to prove a version of the Prékopa-Leindler inequality on the sphere, remarking that a similar version can be obtained for hyperbolic space. These results are generalized in a remarkable paper [42] by Cordero-Erausquin, McCann, and Schmuckenschläger, who establish a beautiful Riemannian version of Theorem 10.2.

19.14. **Further applications.** The Brunn-Minkowski inequality has been used in the study of crystals. A crystal in contact with its melt (or a liquid in contact with its vapor) is modeled by a bounded Borel subset M of $\mathbb{R}^n$ of finite surface area and fixed volume. (We shall ignore measure-theoretic subtleties in this description.) The *surface energy* is given by

$$F(M) = \int_{\partial M} f(u_x)\, dx,$$

where u_x is the outer unit normal to M at x and f is a nonnegative function on S^{n-1} representing the surface tension, assumed known by experiment or theory. By the Gibbs-Curie principle, the equilibrium shape of such a crystal minimizes this surface energy among all sets of the same volume. This shape is called the *Wulff shape.* For a soapy liquid drop in air, f is a constant (we are neglecting external potentials such as gravity) and the Wulff shape is a ball, by the isoperimetric inequality. For crystals, however, f will generally reflect certain preferred directions. In 1901, Wulff gave a construction of the Wulff shape W:

$$W = \cap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq f(u)\};$$

each set in the intersection is a half-space containing the origin with bounding hyperplane orthogonal to u and containing the point $f(u)u$ at distance $f(u)$ from the origin. The Brunn-Minkowski inequality can be used to prove that, up to translation, W is the unique shape among all with the same volume for which F is minimum; see, for example, [143, Theorem 1.1]. This was done first by A. Dinghas in 1943 for convex polygons and polyhedra and then by various people in greater generality. In particular, Busemann [38] solved the problem when f is continuous, and Fonseca [62] and Fonseca and Müller [63] extend the results to include sets M of finite perimeter in $\mathbb{R}^n$. Good introductions with more details and references are provided by Taylor [143] and McCann [115].

In fact, McCann [115] also proves more general results that incorporate a convex external potential, by a technique developed in his paper [114] on interacting gases. A gas of particles in $\mathbb{R}^n$ is modeled by a nonnegative mass density $\rho(x)$ of total integral 1, that is, a probability density on $\mathbb{R}^n$, or, equivalently, by an absolutely continuous probability measure in $\mathbb{R}^n$. To each state corresponds an energy

$$\begin{aligned} E(\rho) &= U(\rho) + \frac{G(\rho)}{2} \\ &= \int_{\mathbb{R}^n} A(\rho(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} V(x-y)\,d\rho(x)\,d\rho(y). \end{aligned}$$

Here U represents the internal energy with A a convex function defined in terms of the pressure, and $G(\rho)/2$ is the potential energy defined by a strictly convex interaction potential V. The problem is that $E(\rho)$ is not generally convex, making it nontrivial to prove the uniqueness of an energy minimizer. McCann gets around this by defining for each pair ρ, ρ' of probability densities on $\mathbb{R}^n$ and $0 < t < 1$ an interpolant probability density ρ_t such that

$$U(\rho_t) \le (1-t)U(\rho) + tU(\rho') \tag{75}$$

(and similarly for G and hence for E). McCann calls (75) the *displacement convexity* of U; ρ_t is *not* $(1-t)\rho + t\rho'$, but rather is defined in the natural way by means of the Brenier map that transports ρ to ρ' (see the last paragraph of Section 17). McCann is also able to recover the Brunn-Minkowski inequality from (75) by taking $A(\rho) = -\rho^{(n-1)/n}$ and ρ and ρ' to be the densities corresponding to the uniform probability measures on the two sets.

Next we turn to applications to diffusion equations. Let V be a nonnegative continuous potential defined on a convex domain C in $\mathbb{R}^n$ and consider the diffusion equation

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}\triangle\psi - V(x)\psi(x,t) \tag{76}$$

with zero Dirichlet boundary condition (i.e., ψ tends to zero as x approaches the boundary of C for each fixed t). Denote by $f(t,x,y)$ the fundamental solution of (76); that is, $\psi(t,x) = f(t,x,y)$ satisfies (76) and its boundary condition, and

$$\lim_{t\to 0+} f(t,x,y) = \delta(x-y).$$

For example, if $V = 0$ and $C = \mathbb{R}^n$, then

$$f(t,x,y) = (2\pi t)^{-n/2} e^{-|x-y|^2/2t}.$$

Brascamp and Lieb [35] proved that if V is convex, then $f(t,x,y)$ is log concave on C^2. This is an application of the Prékopa-Leindler inequality, via Theorem 11.3 with $p = 0$; basically, it is shown that f is given as a pointwise limit of convolutions of log concave functions (Gaussians or $\exp(-tV(x))$). Borell [30] uses a version of Theorem 10.2 to show that the stronger assumption that V is $-1/2$-concave implies that $t\log(t^n f(t^2,x,y))$ is concave on $\mathbb{R}_+ \times C^2$. In a further study, Borell [32] generalizes all of these results (and the Prékopa-Leindler inequality) by considering potentials $V(\sigma, x)$ that depend also on a parameter σ.

Another rich area of applications surrounds the *logarithmic Sobolev inequality* proved by Gross [73]:

$$\mathrm{Ent}_{\gamma_n}(f) \le \frac{1}{2} I_{\gamma_n}(f), \tag{77}$$

where f is a suitably smooth nonnegative function on $\mathbb{R}^n$, γ_n is the Gauss measure defined in the previous subsection,

$$\mathrm{Ent}_{\gamma_n}(f) = \int_{\mathbb{R}^n} f \log f \, d\gamma_n - \left(\int_{\mathbb{R}^n} f \, d\gamma_n \right) \left(\int_{\mathbb{R}^n} \log f \, d\gamma_n \right),$$

and

$$I_{\gamma_n}(f) = \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, d\gamma_n.$$

Note (see Section 18) that $\mathrm{Ent}_{\gamma_n}(f)$ and $I_{\gamma_n}(f)$ are essentially the negative entropy $-h_1(f)$ and Fisher information, respectively, of f, defined with respect to Gauss measure. Bobkov and Ledoux [25] derive (77) from the Prékopa-Leindler inequality (the "Brascamp-Lieb" in the title of [25] refers to a different inequality of Brascamp and Lieb proved in [35]). Cordero-Erausquin [40] proves (77) directly using the transportation of mass idea seen in action above.

McCann's displacement convexity (75) plays an essential role in very recent work involving several of the above topics. Otto [125] observed that various diffusion equations can be viewed as gradient flows in the space of probability measures with the Wasserstein metric (formally, at least, an infinite-dimensional Riemannian structure). McCann's interpolation using the Brenier map gives the geodesics in this space, and Otto uses the displacement convexity to derive rates of convergence to equilibrium. The same ideas are utilized by Otto and Villani [126], who find a new proof of an inequality of Talagrand for the Wasserstein distance between two probability measures in an n-dimensional Riemannian manifold, and show that Talagrand's inequality is very closely related to the logarithmic Sobolev inequality (77). The interested reader may also consult Ledoux's survey [85].

The Brunn-Minkowski inequality was used by Firey [61] in an investigation of the shapes of worn stones, related to the $p = 0$ version of the L^p-Minkowski problem (see Section 19.4). There is a connection here (as well as for the topic of shapes of crystals described above) with an active area concerning curvature-driven flows; see, in particular, Andrews' solution [3] of a conjecture of Firey in [61]. Borell [31] applies Theorem 11.2 and his Brunn-Minkowski inequality in Gauss space (see the previous subsection) to option pricing, assuming that underlying stock prices are governed by a joint Brownian motion. Kannan, Lovász, and Simonovits [81] obtain some inequalities involving log-concave functions by means of a "localization lemma" that reduces certain inequalities involving integrals over convex bodies in $\mathbb{R}^n$ to integral inequalities over "infinitesimal truncated cones"—line segments with associated linear functions—and hence to inequalities in a single variable. The proof of this localization lemma uses the Brunn-Minkowski inequality; see [93, Lemma 2.5], where an application to the algorithmic computation of volume is discussed. Other applications of the Brunn-Minkowski inequality include elliptic partial differential equations [7] and combinatorics [80].

References

[1] S. Alesker, S. Dar, and V. Milman, *A remarkable measure preserving diffeomorphism between two convex bodies in* $\mathbb{R}^n$, Geom. Dedicata **74** (1999), 201–212.

[2] T. W. Anderson, *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. **6** (1955), 170–176.

[3] B. Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), 151–161.

[4] J. Arias-de Reyna, Keith Ball, and Rafael Villa, *Concentration of the distance in finite dimensional normed spaces*, Mathematika **45** (1998), 245–252.
[5] H. Bahn and P. Ehrlich, *A Brunn-Minkowski type theorem on the Minkowski spacetime*, Canad. J. Math. **51** (1999), 449–469.
[6] R. Baierlein, *Atoms and Information Theory*, W. H. Freeman and Company, San Francisco, 1971.
[7] I. J. Bakelman, *Convex Analysis and Nonlinear Geometric Elliptic Equations*, Springer, Berlin, 1994.
[8] K. M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbf{R}^n$, Studia Math. **88** (1988), 69–84.
[9] ———, *Volumes of sections of cubes and related matters*, Geometric Aspects of Functional Analysis, ed. by J. Lindenstrauss and V. D. Milman, Lecture Notes in Mathematics 1376, Springer, Heidelberg, 1989, pp. 251–60.
[10] ———, *Shadows of convex bodies*, Trans. Amer. Math. Soc. **327** (1991), 891–901.
[11] ———, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) **44** (1991), 351–359.
[12] ———, *Ellipsoids of maximal volume in convex bodies*, Geom. Dedicata **41** (1992), 241–250.
[13] ———, *An elementary introduction to modern convex geometry*, Flavors of Geometry, ed. by Silvio Levy, Cambridge University Press, New York, 1997, pp. 1–58.
[14] F. Barthe, *Inégalités de Brascamp-Lieb et convexité*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 885–888.
[15] ———, *Inégalités Fonctionelles et Géométriques Obtenues par Transport des Mesures*, PhD thesis, Université de Marne-la-Vallée, Paris, 1997.
[16] ———, *An extremal property of the mean width of the simplex*, Math. Ann. **310** (1998), 685–693.
[17] ———, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 335–361.
[18] ———, *Optimal Young's inequality and its converse: a simple proof*, Geom. Funct. Anal. **8** (1998), 234–242.
[19] ———, *Restricted Prékopa-Leindler inequality*, Pacific J. Math **189** (1999), 211–222.
[20] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1965.
[21] W. Beckner, *Inequalities in Fourier analysis*, Ann. of Math. **102** (1975), 159–182.
[22] F. Behrend, *Über die kleinste umbeschriebene und die gröste einbeschriebene Ellipse eines konvexen Bereichs*, Math. Ann. **115** (1938), 379–411.
[23] G. Bianchi, *Determining convex bodies with piecewise* C^2 *boundary from their covariogram*, preprint.
[24] N. M. Blachman, *The convolution inequality for entropy powers*, IEEE Trans. Information Theory **11** (1965), 267–271.
[25] S. G. Bobkov and M. Ledoux, *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities*, Geom. Funct. Anal. **10** (2000), 1028–1052.
[26] B. Bollobás and I. Leader, *Sums in the grid*, Discrete Math. **162** (1996), 31–48.
[27] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207–216.
[28] ———, *Convex set functions in d-space*, Period. Math. Hungar. **6** (1975), 111–136.
[29] ———, *Capacitary inequalities of the Brunn-Minkowski type*, Math. Ann. **263** (1983), 179–184.
[30] ———, *Geometric properties of some familiar diffusions in* $\mathbb{R}^n$, Ann. of Probab. **21** (1993), 482–489.
[31] ———, *Geometric inequalities in option pricing*, Convex Geometric Analysis, ed. by K. M. Ball and V. Milman, Cambridge University Press, Cambridge, 1999, pp. 29–51.
[32] ———, *Diffusion equations and geometric inequalities*, Potential Anal. **12** (2000), 49–71.
[33] H. J. Brascamp and E. H. Lieb, *Some inequalities for Gaussian measures and the long-range order of one-dimensional plasma*, Functional Integration and its Applications, ed. by A. M. Arthurs, Clarendon Press, Oxford, 1975, pp. 1–14.
[34] ———, *Best constants in Young's inequality, its converse, and its generalization to more than three functions*, Adv. Math. **20** (1976), 151–173.
[35] ———, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Anal. **22** (1976), 366–389.
[36] V. V. Buldygin and A. B. Kharazishvili, *Geometric Aspects of Probability Theory and Mathematical Statistics*, Kluwer, Dordrecht, 2000. Russian original: 1985.
[37] Y. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer, New York, 1988. Russian original: 1980.
[38] H. Busemann, *The isoperimetric problem for Minkowski area*, Amer. J. Math. **71** (1949), 743–762.
[39] L. A. Caffarelli, D. Jerison, and E. Lieb, *On the case of equality in the Brunn-Minkowski inequality for capacity*, Adv. Math. **117** (1996), 193–207.

[40] D. Cordero-Erausquin, *Some applications of mass transport to Gaussian type inequalities*, Arch. Rational Mech. Anal., to appear.
[41] ———, *Inégalité de Prékopa-Leindler sur la sphère*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), 789–792.
[42] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math., to appear.
[43] M. H. M. Costa and T. M. Cover, *On the similarity of the entropy power inequality and the Brunn-Minkowski inequality*, IEEE Trans. Information Theory **30** (1984), 837–839.
[44] S. Dancs and B. Uhrin, *On a class of integral inequalities and their measure-theoretic consequences*, J. Math. Anal. Appl. **74** (1980), 388–400.
[45] ———, *On the conditions of equality in an integral inequality*, Publ. Math. (Debrecen) **29** (1982), 117–132.
[46] S. Dar, *A Brunn-Minkowski-type inequality*, Geom. Dedicata **77** (1999), 1–9.
[47] S. Das Gupta, *Brunn-Minkowski and its aftermath*, J. Multivariate Analysis **10** (1980), 296–318.
[48] A. Dembo, T. M. Cover, and J. A. Thomas, *Information theoretic inequalities*, IEEE Trans. Information Theory **37** (1991), 1501–1518.
[49] S. Dharmadhikari and K. Joag-Dev, *Unimodality, convexity, and applications*, Academic Press, New York, 1988.
[50] A. Dinghas, *Über eine Klasse superadditiver Mengenfunktionale von Brunn-Minkowski-Lusternikschem Typus*, Math. Zeit. **68** (1957), 111–125.
[51] R. M. Dudley, *Real Analysis and Probability*, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1989.
[52] ———, *Metric marginal problems for set-valued or non-measurable variables*, Probab. Theory Relat. Fields **100** (1994), 175–189.
[53] H. G. Eggleston, *Convexity*, Cambridge University Press, Cambridge, 1958.
[54] A. Ehrhard, *Symétrisation dans l'espace de Gauss*, Math. Scand. **53** (1983), 281–301.
[55] ———, *Élements extrémaux pour les inégalités de Brunn-Minkowski gaussiennes*, Ann. Inst. H. Poincaré Probab. Statist. **22** (1986), 149–168.
[56] P. Erdös and A. H. Stone, *On the sum of two Borel sets*, Proc. Amer. Math. Soc. **25** (1970), 304–306.
[57] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.
[58] H. Federer, *Geometric Measure Theory*, Springer, New York, 1969.
[59] W. J. Firey, *Polar means and a dual to the Brunn-Minkowski theorem*, Canad. J. Math. **13** (1961), 444–453.
[60] ———, *p-means of convex bodies*, Math. Scand. **10** (1962), 17–24.
[61] ———, *Shapes of worn stones*, Mathematika **21** (1974), 1–11.
[62] I. Fonseca, *The Wulff theorem revisited*, Proc. Roy. Soc. London Sect. A **432** (1991), 125–145.
[63] I. Fonseca and S. Müller, *A uniqueness proof for the Wulff theorem*, Proc. Roy. Soc. Edinburgh Sect. A **119** (1991), 125–136.
[64] B. R. Frieden, *Physics from Fisher Information: a Unification*, Cambridge University Press, New York, 1999.
[65] R. J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435–445.
[66] ———, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. **140** (1994), 435–447.
[67] ———, *Geometric Tomography*, Cambridge University Press, New York, 1995.
[68] R. J. Gardner and P. Gronchi, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc. **353** (2001), 3995–4024.
[69] R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytical solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. **149** (1999), 691–703.
[70] R. J. Gardner and G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math. **120** (1998), 505–528.
[71] H. Groemer, *Stability of geometric inequalities*, Handbook of Convexity, ed. by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, 1993, pp. 125–150.
[72] M. Gromov, *Convex sets and Kähler manifolds*, Advances in Differential Geometry and Topology, World Scientific Publishing, Teaneck, NJ, 1990, pp. 1–38.
[73] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.

[74] O.-G. Guleryuz, E. Lutwak, D. Yang, and G. Zhang, *Information theoretic inequalities for contoured probability distributions*, preprint.

[75] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[76] H. Hadwiger and D. Ohmann, *Brunn-Minkowskischer Satz und Isoperimetrie*, Math. Zeit. **66** (1956), 1–8.

[77] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1959.

[78] R. Henstock and A. M. Macbeath, *On the measure of sum sets, I. The theorems of Brunn, Minkowski and Lusternik*, Proc. London Math. Soc. **3** (1953), 182–194.

[79] D. Jerison, *A Minkowski problem for electrostatic capacity*, Acta Math. **176** (1996), 1–47.

[80] J. Kahn and N. Linial, *Balancing extensions via Brunn-Minkowski*, Combinatorica **11** (1991), 363–368.

[81] R. Kannan, L. Lovász, and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Math. **13** (1995), 541–559.

[82] J. F. C. Kingman and S. J. Taylor, *Introduction to Measure and Probability*, Cambridge University Press, Cambridge, 1973.

[83] H. Knothe, *Contributions to the theory of convex bodies*, Michigan Math. J. **4** (1957), 39–52.

[84] R. Latała, *A note on the Ehrhard inequality*, Studia Math. **118** (1996), 169–174.

[85] M. Ledoux, *Concentration of measure and logarithmic Sobolev inequalities*, Séminaire de Probabilités, ed. by J. Azéma, M. Émery, M. Ledoux, and M. Yor, Lecture Notes in Mathematics 1709, Springer, Berlin, 1999, pp. 120–216.

[86] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer, New York, 1991.

[87] L. Leindler, *On a certain converse of Hölder's inequality. II*, Acta Sci. Math. (Szeged) **33** (1972), 217–223.

[88] E. H. Lieb, *Proof of an entropy conjecture of Wehrl*, Commun. Math. Phys. **62** (1978), 35–41.

[89] ———, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. **18** (1983), 349–374.

[90] ———, *Gaussian kernels have only Gaussian maximizers*, Invent. Math. **102** (1990), 179–208.

[91] E. H. Lieb and M. Loss, *Analysis*, Second edition, American Mathematical Society, Providence, Rhode Island, 2001.

[92] J. Lindenstrauss and V. D. Milman, *The local theory of normed spaces and its applications to convexity*, Handbook of Convex Geometry, ed. by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, 1993, pp. 1149–1220.

[93] L. Lovász and M. Simonovits, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms **4** (1993), 359–412.

[94] L. A. Lusternik, *Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen*, C. R. Acad. Sci. URSS **8** (1935), 55–58.

[95] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.

[96] ———, *Width-integrals of convex bodies*, Proc. Amer. Math. Soc. **53** (1975), 435–439.

[97] ———, *A general isepiphanic inequality*, Proc. Amer. Math. Soc. **90** (1984), 415–421.

[98] ———, *Volume of mixed bodies*, Trans. Amer. Math. Soc. **294** (1986), 487–500.

[99] ———, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[100] ———, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. (3) **60** (1990), 365–391.

[101] ———, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowksi problem*, J. Diff. Geom. **38** (1993), 131–150.

[102] ———, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), 901–916.

[103] ———, *Selected affine isoperimetric inequalities*, Handbook of Convex Geometry, ed. by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, 1993, pp. 151–176.

[104] ———, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[105] E. Lutwak, D. Yang, and G. Zhang, *The Brunn-Minkowski-Firey inequality for non-convex sets*, preprint.

[106] ———, *The Cramer-Rao inequality for star bodies*, Duke Math. J., to appear.

[107] ———, *On the L_p-Minkowski problem*, preprint.

[108] ———, *A new ellipsoid associated with convex bodies*, Duke Math. J. **104** (2000), 375–90.

[109] ———, *On L_p affine isoperimetric inequalities*, J. Diff. Geom. **56** (2000), 111–132.

[110] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Diff. Geom. **47** (1997), 1–16.

[111] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.
[112] B. Maurey, *Some deviation inequalities*, Geom. Funct. Anal. **1** (1991), 188–197.
[113] R. J. McCann, *A Convexity Theory for Interacting Gases and Equilibrium Crystals*, Ph. D. dissertation, Princeton University, 1994.
[114] ———, *A convexity principle for interacting gases*, Adv. Math. **128** (1997), 153–179.
[115] ———, *Equilibrium shapes for planar crystals in an external field*, Comm. Math. Phys. **195** (1998), 699–723.
[116] P. McMullen, *New combinations of convex sets*, Geom. Dedicata **78** (1999), 1–19.
[117] J. Mecke and A. Schwella, *Inequalities in the sense of Brunn-Minkowski, Vitale for random convex bodies*, preprint.
[118] M. Meyer, *Maximal hyperplane sections of convex bodies*, Mathematika **46** (1999), 131–136.
[119] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric Aspects of Functional Analysis, ed. by J. Lindenstrauss and V. D. Milman, Lecture Notes in Mathematics 1376, Springer, Heidelberg, 1989, pp. 64–104.
[120] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Springer (Lecture Notes in Mathematics 1200), Berlin, 1986.
[121] W. Nagel, *Orientation-dependent chord length distributions characterize convex polygons*, J. Appl. Prob. **30** (1993), 730–736.
[122] M. B. Nathanson, *Additive Number Theory - Inverse Problems and the Geometry of Sumsets*, Springer, New York, 1996.
[123] A. Okounkov, *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411.
[124] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.
[125] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations, to appear.
[126] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** (2000), 361–400.
[127] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge University Press, Cambridge, 1989.
[128] A. Prékopa, *Logarithmic concave measures with application to stochastic programming*, Acta Sci. Math. (Szeged) **32** (1971), 301–315.
[129] ———, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1975), 335–343.
[130] Y. Rinott, *On convexity of measures*, Ann. Probab. **4** (1976), 1020–1026.
[131] I. Z. Ruzsa, *The Brunn-Minkowski inequality and nonconvex sets*, Geom. Dedicata **67** (1997), 337–348.
[132] M. Schmitt, *On two inverse problems in mathematical morphology*, Mathematical Morphology in Image Processing, ed. by E. R. Dougherty, Marcel Dekker, New York, 1993, pp. 151–169.
[133] M. Schmuckenschläger, *An extremal property of the regular simplex*, Convex Geometric Analysis, ed. by K. M. Ball and V. Milman, Cambridge University Press, New York, 1999, pp. 199–202.
[134] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.
[135] J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, London, 1982.
[136] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 623–656. Can be downloaded at `http://www.math.washington.edu/~hillman/Entropy/infcode.html`.
[137] W. Sierpiński, *Sur la question de la mesurabilité de la base de M. Hamel*, Fund. Math. **1** (1920), 105–111.
[138] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control **2** (1959), 101–112.
[139] A. Stancu, *On the number of solutions to the discrete two dimensional L_0-Minkowski problem*, preprint.
[140] D. Stoyan, W. S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*, Akademie-Verlag, Berlin, 1987.
[141] V. N. Sudakov and B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures*, J. Soviet Math. **9** (1978), 9–18. Translated from Zap. Nauch. Sem. L.O.M.I. **41** (1974), 14–24.
[142] S. J. Szarek and D. Voiculescu, *Volumes of restricted Minkowski sums and the free analogue of the entropy power inequality*, Comm. Math. Phys. **178** (1996), 563–570.
[143] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc. **84** (1978), 568–588.

[144] Y. L. Tong, *Probability inequalities in multivariate distributions*, Academic Press, New York, 1980.
[145] N. S. Trudinger, *Isoperimetric inequalities for quermassintegrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 411–425.
[146] B. Uhrin, *Extensions and sharpenings of Brunn-Minkowski and Bonnesen inequalities*, Intuitive Geometry, Siófok, 1985, ed. by K. Böröczky and G. Fejes Tóth, Coll. Math. Soc. János Bolyai **48**, North-Holland, Amsterdam, 1987, pp. 551–571.
[147] ———, *Curvilinear extensions of the Brunn-Minkowski-Lusternik inequality*, Adv. Math. **109** (1994), 288–312.
[148] R. A. Vitale, *The Brunn-Minkowski inequality for random sets*, J. Multivariate Analysis **33** (1990), 286–293.
[149] ———, *The translative expectation of a random set*, J. Math. Anal. Appl. **160** (1991), 556–562.
[150] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.
[151] Gaoyong Zhang, *Intersection bodies and the Busemann-Petty inequalities in* $\mathbb{R}^4$, Ann. of Math. **140** (1994), 331–346.
[152] ———, *The affine Sobolev inequality*, J. Diff. Geom. **53** (1999), 183–202.
[153] ———, *A positive answer to the Busemann-Petty problem in four dimensions*, Ann. of Math. **149** (1999), 535–543.

Department of Mathematics, Western Washington University, Bellingham, WA 98225-9063
E-mail address: `gardner@baker.math.wwu.edu`