Proof of estimate (7) in *Reconstruction of convex bodies from brightness functions*, by R. J. Gardner and Peyman Milanfar.

(**Note:** No attempt is made here to obtain the best-possible such estimate. It is enough for our purposes to establish a bound that is polynomial in R_0 and r_0^{-1}.)

The following result is stated and proved by Groemer [3, Theorem 5.5.7]: *For origin-symmetric convex bodies K and L in $\mathbb{R}^d$ with $r_0B \subset K, L \subset R_0B$ and $\alpha \in (0, 2/(d(d+4))$,*

$$\delta(K,L) \leq c_d(\alpha, R_0, r_0)\delta(\Pi K, \Pi L)^\alpha.$$

While studying the proof, we shall use Groemer's notation, except in using b_K for the brightness function and V instead of his v for volume.

At the top of [3, p. 232], it is stated that

$$|V_1(K,L) - V(K)| \leq \max\{\eta_d(\alpha, R_0), 2\kappa_d R_0^d \epsilon_d(\alpha)^{-d\alpha}\}|b_K(u) - b_L(u)|^{d\alpha}. \tag{1}$$

From [3, Lemma 5.5.10], specifically by comparing [3, (5.5.20)] with [3, (5.5.26)], we get

$$\eta_d(\alpha, R_0) = \theta_d(R_0)\left(\frac{2(1-d\alpha)}{d+2} + 1\right), \tag{2}$$

and by comparing [3, (5.5.25)] with the previous unnumbered displayed inequality, and using the estimates for $s(K)$, $s(L)$, and $\Lambda(F)$ after it, we also have

$$\theta_d(R_0) \leq \frac{2^{d+1}\sigma_{d-1}}{d\sigma_d}2\sigma_d R_0^d + const(d) = O(R_0^d). \tag{3}$$

Putting (1), (2), and (3) together, we obtain

$$|V_1(K,L) - V(K)| = O(R_0^d)|b_K(u) - b_L(u)|^{d\alpha}. \tag{4}$$

Now (4) and [3, (5.5.27)] show that

$$\delta(K,L) = O(R_0)\mu_d(R_0, r_0)\delta(\Pi K, \Pi L)^\alpha. \tag{5}$$

Next, to estimate $\mu_d(R_0, r_0)$, we have from [3, p. 233, lines 8 and 9], using the fact that $m \leq 1$,

$$\mu_d(R_0, r_0) = V(K)^{1/d}\left(b_d(R_0, r_0)^{1/d} + e_d(r_0)R_0(\kappa_d r_0)^{-1/d}\right). \tag{6}$$

Here $e_d(r_0)$ can be estimated from the expressions on [3, p. 232]. Beginning with

$$\lambda - 1 \leq \frac{m}{V(K)},$$

we get from

$$\frac{1}{\lambda} - 1 \leq \frac{m}{V(L)}$$

that

$$1 - \lambda \leq \frac{\lambda m}{V(L)} \leq \left(1 + \frac{m}{V(K)}\right)\frac{m}{V(L)}.$$

Using $m \leq 1$ and

$$V(K), V(L) \geq \kappa_d r_0^d,$$

this gives

$$|\lambda - 1| \le e_d(r_0)m,$$

where

$$e_d(r_0) = \left(1 + (\kappa_d r_0^d)^{-1}\right)(\kappa_d r_0^d)^{-1}. \tag{7}$$

So $e_d(r_0) = O(r_0^{-2d})$ if $r_0 \le 1$. From [3, p. 232, lines -4 and -3], we have

$$b_d(R_0, r_0) = k_d(R_0, r_0)\left(1 + \kappa_d R_0^d e_d(r_0)\right) = O(R_0^d r_0^{-2d}) k_d(R_0, r_0)$$

if $r_0 \le 1$. Using this expression, (6), (7), and $V(K) \le \kappa_n R_0^d$, we get

$$\mu_d(R_0, r_0) = O(R_0^2 r_0^{-2})\left(k_d(R_0, r_0)^{1/d} + O(r_0^{-2d+1})\right). \tag{8}$$

It remains to estimate $k_d(R_0, r_0)$. This appears in [3, (2.5.13)], which implies that for origin-symmetric convex bodies K and L in $\mathbb{R}^d$ with $r_0 B \subset K, L \subset R_0 B$,

$$\delta(K', L') \le k_d(R_0, r_0)\left(V_1(K, L) - V(K)^{(d-1)/d} V(L)^{1/d}\right), \tag{9}$$

where $K' = (1/V(K))K$ and $L' = (1/V(L))L$.

The estimate (9) was proved by Diskant, and unfortunately Groemer only refers to Diskant's papers for the proof. Therefore we have to study these papers and deal with his different notation. We will compromise by working with origin-symmetric convex bodies K and L in $\mathbb{R}^d$ with $r_0 B \subset K, L \subset R_0 B$, but otherwise our notation will be quite compatible with Diskant's.

Diskant's [2, Lemma 3], for origin-symmetric convex bodies, is the following statement. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^d$ with $r_0 B \subset K, L \subset R_0 B$, and let*

$$\Delta(K, L) = V_1(K, L)^d - V(K)^{d-1} V(L).$$

There is an $\varepsilon_0 > 0$ such that if $\Delta(K, L) < \varepsilon_0$, and $V(K) = V(L)$, then

$$\delta(K, L) \le C \Delta(K, L)^{1/d}. \tag{10}$$

Here ε_0 and C depend only on d, r_0, and R_0.

In what follows, we shall assume that $r_0 < 1$ and $R_0 > 1$, and often use the estimates

$$\kappa_d r_0^d \le V(K), V(L), V_1(K, L) \le \kappa_d R_0^d. \tag{11}$$

First we must match Diskant's statement (10) with Groemer's (9) . Let $K' = (1/V(K))K$ and $L' = (1/V(L))L$. Then $V(K') = V(L') = 1$, so (10) applied to K' and L' implies that

$$\begin{aligned}
\delta(K', L')^d &\leq C^d \left(V_1(K', L')^d - 1\right) \\
&= C^d \left(\frac{V_1(K, L)^d}{V(K)^{d-1}V(L)} - 1\right) \\
&\leq \left(\frac{C}{\kappa_d r_0^d}\right)^d \left(V_1(K, L)^d - V(K)^{d-1}V(L)\right) \\
&\leq \left(\frac{C}{\kappa_d r_0^d}\right)^d \left(V_1(K, L) - V(K)^{\frac{d-1}{d}} V(L)^{\frac{1}{d}}\right) \sum_{i=0}^{d-1} V_1(K, L)^i \left(V(K)^{\frac{d-1}{d}} V(L)^{\frac{1}{d}}\right)^{d-1-i} \\
&\leq \left(\frac{C}{\kappa_d r_0^d}\right)^d \left(V_1(K, L) - V(K)^{\frac{d-1}{d}} V(L)^{\frac{1}{d}}\right) d(\kappa_d R_0^d)^{d-1}.
\end{aligned}$$

Comparing this with (9), we see that

$$k_d(R_0, r_0) = \left(\left(\frac{C}{\kappa_d r_0^d}\right)^d d(\kappa_d R_0^d)^{d-1}\right)^{1/d} = O\left(\frac{R_0^{d-1}}{r_0^d}\right) C. \tag{12}$$

Next we focus on Diskant's proof of (10). We remind the reader of his assumption that $V(K) = V(L)$, and begin by obtaining a more explicit form of [2, Lemma 2]. In the proof of [2, Lemma 2], we find

$$\Delta(K, L) \geq V(K)^{\frac{d(d-2)}{d-1}} \left(V_1(L, K)^{\frac{d}{d-1}} - V(L)V(K)^{\frac{1}{d-1}}\right). \tag{13}$$

Using

$$\begin{aligned}
\Delta(L, K) &= V_1(L, K)^d - V(L)^{d-1}V(K) \\
&= \left(V_1(L, K)^{\frac{d}{d-1}} - V(L)V(K)^{\frac{1}{d-1}}\right) \sum_{i=0}^{d-2} V_1(L, K)^{\frac{id}{d-1}} \left(V(L)V(K)^{\frac{1}{d-1}}\right)^{d-2-i},
\end{aligned}$$

we obtain from (13) that

$$\Delta(K, L) \geq \frac{(\kappa_d r_0^d)^{\frac{d(d-2)}{d-1}}}{(d-1)(\kappa_d R_0^d)^{\frac{d(d-2)}{d-1}}} \Delta(L, K).$$

This gives

$$\Delta(L, K) \leq \overline{C}\Delta(K, L), \tag{14}$$

where

$$\overline{C} = O\left(\frac{R_0^{d^2}}{r_0^{d^2}}\right). \tag{15}$$

In Diskant's earlier work [1], he defines

$$q(K, L) = \max\{\lambda : \lambda L \subset K\}$$

and shows in [1, Theorem 1] that

$$q = q(K,L) \geq \left(\frac{V_1(K,L)}{V(L)}\right)^{\frac{1}{d-1}} - \frac{\left(V_1(K,L)^{\frac{d}{d-1}} - V(K)V(L)^{\frac{1}{d-1}}\right)^{\frac{1}{d}}}{V(L)^{\frac{1}{d-1}}}.$$

This estimate is used in the proof of [1, Theorem 2] (compare the first displayed set of inequalities in this proof) to show that

$$q \geq 1 - \frac{\Delta(K,L)^{1/d}}{V(L)^{\frac{1}{d-1}} V_1(K,L)^{\frac{d-2}{d-1}}}.$$

From this and (11) we obtain

$$q \geq 1 - \frac{1}{\kappa_d r_0^d}\Delta(K,L)^{1/d}. \tag{16}$$

(This is a slightly different constant from that obtained by Diskant.) Interchanging K and L and using (14), we obtain by the same method,

$$q_1 = q(L,K) \geq 1 - \frac{1}{\kappa_d r_0^d}\Delta(L,K)^{1/d} \geq 1 - \frac{\overline{C}^{1/d}}{\kappa_d r_0^d}\Delta(K,L)^{1/d}. \tag{17}$$

Also in the proof of [1, Theorem 2], Diskant notes that $q, q_1 \leq 1$. In [1, Lemma 4], it is proved that if $q, q_1 \leq 1$, then

$$\delta(K,L) \leq \frac{2R_0}{q_1}(1 - qq_1) = 2R_0\left(\frac{1}{q_1} - q\right).$$

Therefore, using (16) and (17), we obtain

$$\delta(K,L) \leq 2R_0\left(\frac{1}{1 - \frac{\overline{C}^{1/d}}{\kappa_d r_0^d}\Delta(K,L)^{1/d}} - 1 + \frac{1}{\kappa_d r_0^d}\Delta(K,L)^{1/d}\right) \leq C\Delta(K,L)^{1/d},$$

where, by (15),

$$C = \frac{2R_0}{\kappa_d r_0^d}(1 + \overline{C}^{1/d}) = O\left(\frac{R_0^{d+1}}{r_0^{2d}}\right). \tag{18}$$

Thus we have obtained (10) without the need for the restriction $\Delta(K,L) < \varepsilon_0$.

Putting together (5), (8), (12), and (18), we obtain

$$\delta(K,L) = O\left(\frac{R_0^5}{r_0^{2d+1}}\right)\delta(\Pi K, \Pi L)^{\alpha}.$$

References

[1] V. I. Diskant, *Bounds for the discrepancy between convex bodies in terms of the isoperimetric difference*, Siberian Math. J. **13** (1973), 529–532.

[2] V. I. Diskant, *Stability of the solution of the Minkowski equation*, Siberian Math. J. **14** (1974), 466–469.

[3] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*, Cambridge University Press, New York, 1996.